

Quantum Distortion Gravity: Classical Foundations, Gauge Structure and the Einstein–Hilbert Flow with QDG-Like Matter

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Abstract

Quantum Distortion Gravity (QDG) is a metric-affine proposal in which the affine connection is an independent field and Weyl invariance is gauged with a dilaton compensator. This paper collects what can be stated and checked about it. *Classically*: the irreducible decomposition $64 = 4 + 12 + 48$ of the distortion tensor and its quadratic invariants (with the convention for the cyclic invariant stated explicitly and the effect of the transposed one listed); a no-go statement showing that no bare quadratic invariant of the distortion is Weyl invariant, because of its homogeneous weight, with the full local variation and the family that is invariant under the inhomogeneous shift alone; the Weyl-invariant truncated action with free coefficients and its reduction in unitary gauge to Einstein gravity plus two massive vector fields. *Constraint and gauge structure*: eight propagating bosonic degrees of freedom at quadratic order (a linearised count, tree-level unitarity only, with a C^2 term generated at one loop, so no ghost-freedom claim); BRST nilpotency and the classical master equation of Weyl $\rtimes$ Diff; the status of the Weyl anomaly, whose coefficients for the gauged symmetry are *not* known. *Quantum flow*: the Einstein–Hilbert flow from the Wetterich equation (optimised cutoff, harmonic gauge, linear split, S^4) with minimally coupled matter of QDG-like content (n_f Dirac fermions and two Proca fields in the massless limit) reduces to a cubic equation. A non-Gaussian fixed point exists for every $n_f < 10$; at $n_f = 3$ it is at $(\tilde{G}_*, \lambda_*) = (2.73, -0.318)$ with real exponents $(2.21, 0.72)$, exists for nine of nine regulators, but moves strongly with the regulator and with the matter-cutoff convention. A phenomenological flow used in earlier versions is kept, labelled as a model. The Weyl-gauged sector of QDG is not part of the flow, so the paper establishes no statement of asymptotic safety for QDG itself.

Keywords: quantum gravity, metric-affine gravity, Weyl symmetry, BRST–BV quantization, Weyl anomaly, asymptotic safety, functional renormalization group, Proca fields

1 Introduction

Quantum Distortion Gravity (QDG) is a metric-affine framework in which the independent affine connection $\Gamma^\rho_{\mu\nu}$ differs from the Levi-Civita connection $\hat{\Gamma}^\rho_{\mu\nu}$ of the metric by the *distortion* tensor $D^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \hat{\Gamma}^\rho_{\mu\nu}$, and in which local Weyl rescalings are a gauge symmetry compensated by a dilaton ϕ and a Weyl vector V_μ . Its detailed development, with all derivations and scripts, is in the second edition of Ref. [22]. The first version of the present paper [18] presented the construction together with applications (black holes, cosmology, gravitational waves, spectral dimension) and a fixed-point result obtained from a phenomenological flow. This version is deliberately narrower. It keeps the parts that are derived and reproduced by scripts, drops the

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applications (black holes, cosmology, gravitational waves, spectral dimension, swampland constraints), about which this paper makes no claim, replaces the model flow by the flow derived from the Einstein–Hilbert action, and says where the evidence stops.

Contents. Section 2 gives the classical foundations: the decomposition of the distortion, the quadratic invariants and the stated convention for the cyclic invariant I_2 , the Weyl transformation, the no-go statement for bare invariants, the Weyl-invariant truncated action and its unitary-gauge reduction. Section 3 is the constraint analysis at quadratic order. Section 4 is the BRST–BV structure. Section 5 states what is and is not known about the Weyl anomaly. Section 6 derives the Einstein–Hilbert flow with matter of QDG-like content and its fixed point. Section 7 records the phenomenological model flow, labelled as such. Section 8 summarises the status, lists the claims of the first version that are not repeated, and states the open problems.

Logical status. *Proposition* and *Theorem* mean proven within the stated assumptions; *Assumption* denotes an input; *Remark* and *Open Problem* mean not proven. Headings and tables carry tags: [DERIVED] (derived in the paper or in Ref. [22], and reproduced by a script, Appendix A), [INPUT] (taken from a cited source that was checked), [MODEL] (phenomenological, not derived from QDG) and [OPEN] (not established).

Relation to companion papers. Two companion papers study the Einstein–Hilbert flow with QDG-like matter [20] and the effect of additional hidden matter sectors on its fixed point [21]. Section 6 reproduces the part of that analysis that this paper needs, with its own scripts, and does not go beyond it.

2 Classical foundations

2.1 Conventions

Signature $(-, +, +, +)$. The Levi-Civita *tensor* is $\varepsilon^{\sigma\rho\mu\nu} = [\sigma\rho\mu\nu]/\sqrt{-g}$ with $[0123] = +1$, so $\varepsilon^{0123} = +1$ in a Minkowski frame and $\varepsilon_{\sigma\rho\mu\nu} = -\sqrt{-g}[\sigma\rho\mu\nu]$; $\varepsilon^{\tau\rho\mu\nu}\varepsilon_{\sigma\rho\mu\nu} = -6\delta_\sigma^\tau$. Covariant derivative $\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma}V^\sigma$; torsion $T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}$; non-metricity $Q_{\mu\nu\rho} = -\nabla_\mu g_{\nu\rho}$ (the convention of Ref. [11]). Indices of $D_{\rho\mu\nu} = g_{\rho\sigma}D^\sigma_{\mu\nu}$ are raised with $g^{\mu\nu}$. A Weyl rescaling is $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $\omega = \ln \Omega$; a field of weight w transforms as $\Phi \rightarrow \Omega^w \Phi$ (dilaton $w = -1$, Dirac field $w = -\frac{3}{2}$); the Weyl vector transforms as $V_\mu \rightarrow V_\mu - \partial_\mu \omega$, the covariant derivative is $D_\mu \Phi = (\partial_\mu + wV_\mu)\Phi$ (so $D_\mu \phi = (\partial_\mu - V_\mu)\phi$), and $\tilde{R} = \hat{R} - 6\hat{\nabla}_\mu V^\mu - 6V_\mu V^\mu$ with $\tilde{R} \rightarrow \Omega^{-2}\tilde{R}$. The distortion has $64 = 24 + 40$ components (24 of torsion, 40 of non-metricity).

2.2 Irreducible decomposition [DERIVED]

Define the three vector traces and the axial vector of $D_{\rho\mu\nu}$,

$$A_\nu = g^{\rho\mu}D_{\rho\mu\nu}, \quad B_\mu = g^{\rho\nu}D_{\rho\mu\nu}, \quad C_\rho = g^{\mu\nu}D_{\rho\mu\nu}, \quad a^\sigma = \frac{1}{6}\varepsilon^{\sigma\rho\mu\nu}D_{\rho\mu\nu}. \quad (1)$$

With $t_\nu = T^\sigma_{\sigma\nu}$, $N_\mu = Q_{\mu\nu}{}^\nu$ and $\bar{N}_\rho = Q^\mu_{\mu\rho}$ one finds $A_\nu = t_\nu + \frac{1}{2}N_\nu$, $B_\mu = \frac{1}{2}N_\mu$, $C_\rho = -t_\rho + \bar{N}_\rho - \frac{1}{2}N_\rho$, and $a^\sigma = \frac{1}{12}\varepsilon^{\sigma\rho\mu\nu}T_{\rho\mu\nu}$ is a pure torsion quantity (script `partI_II_audit_algebra.py`, residuals $\leq 10^{-15}$).

Proposition 2.1 (Decomposition of the distortion). *Every $D_{\rho\mu\nu}$ can be written uniquely as*

$$D_{\rho\mu\nu} = \varepsilon_{\rho\mu\nu\sigma}a^\sigma + g_{\rho\mu}P_\nu + g_{\rho\nu}Q_\mu + g_{\mu\nu}R_\rho + \hat{q}_{\rho\mu\nu}, \quad (2)$$

with a^σ as in Eq. (1), $P = \frac{1}{18}(5A - B - C)$, $Q = \frac{1}{18}(5B - A - C)$, $R = \frac{1}{18}(5C - A - B)$, and $\hat{q}_{\rho\mu\nu}$ having vanishing traces and vanishing totally antisymmetric part. The component count is $64 = 4 + 12 + 48$.

Proof. Contracting $g_{\rho\mu}P_\nu + g_{\rho\nu}Q_\mu + g_{\mu\nu}R_\rho$ with $g^{\rho\mu}, g^{\rho\nu}, g^{\mu\nu}$ gives $(A, B, C)^T = M(P, Q, R)^T$ with $M = 3 \cdot \mathbb{K} + J$, $J_{ij} = 1$. Since $J^2 = 3J$, $M^{-1} = \frac{1}{3}(\mathbb{K} - \frac{1}{6}J)$, which gives the stated P, Q, R (so M has eigenvalues 6, 3, 3 and is invertible). The term εa is totally antisymmetric and traceless, $D^{(0)} = gP + gQ + gR$ has no totally antisymmetric part, and $\varepsilon^{\tau\rho\mu\nu}\varepsilon_{\rho\mu\nu\sigma} = +6\delta_\sigma^\tau$ gives $D_{[\rho\mu\nu]} = \varepsilon_{\rho\mu\nu\sigma}a^\sigma$. Hence $\hat{q} \equiv D - \varepsilon a - D^{(0)}$ has the three traces and the axial vector equal to zero (existence). If $\varepsilon a + D^{(0)} + \hat{q} = 0$, the totally antisymmetric part gives $a = 0$, the traces give $M(P, Q, R)^T = 0$, hence $P = Q = R = 0$ and $\hat{q} = 0$ (uniqueness). The map $D \mapsto (a, A, B, C)$ has a right inverse, so its rank is $4 + 12 = 16$ and its kernel has dimension $64 - 16 = 48$. $\square$

The script `classical_checks.py` checks Eq. (2) on a random generic Lorentzian metric and a random D (residuals $\leq 10^{-13}$) and the rank 16 of $D \mapsto (a, A, B, C)$.

Remark 2.2 (The 48 is reducible). The tensor $\hat{q}$ is not an irreducible representation. Under $\text{GL}(4)$ the product of three vectors contains the totally symmetric tensor, the totally antisymmetric one and two mixed-symmetry ones; the traceless parts give $48 = 16 + 16 + 16$. Numerically, the quadratic forms $\hat{q}_{\rho\mu\nu}\hat{q}^{\rho\mu\nu}$, $\hat{q}_{\rho\mu\nu}\hat{q}^{\mu\nu\rho}$, $\hat{q}_{\rho\mu\nu}\hat{q}^{\rho\nu\mu}$ restricted to the 48-dimensional space have, relative to $\hat{q}_{\rho\mu\nu}\hat{q}^{\rho\mu\nu}$, the eigenvalues $(+1)^{16}, (-\frac{1}{2})^{32}$ for the cyclic contraction and $(-1)^{16}, (+1)^{32}$ for the transposed one (Euclidean metric; script `classical_checks.py`); the count $4 + 12 + 48$ is unaffected.

2.3 Quadratic invariants and the convention for I_2 [DERIVED]

Contracting $D_{\rho\mu\nu}D_{\alpha\beta\gamma}$ with three inverse metrics gives 15 pairings. Nine contain a trace on each factor and give six products of A, B, C ; the other six are cross contractions, five of them inequivalent under the exchange of the two factors. This gives at most 11 parity-even invariants; they are linearly independent, and with one ε there are six parity-odd ones, the space of Lorentz-invariant quadratic forms on D having dimension $17 = 11 + 6$ (exact rank computations, `partI_II_decomposition.py`). We use

$$I_1 = D_{\rho\mu\nu}D^{\rho\mu\nu}, \quad I_2 = D_{\rho\mu\nu}D^{\mu\nu\rho}, \quad I_3 = C_\rho C^\rho, \quad (3)$$

and, for comparison, $I_5 = D_{\rho\mu\nu}D^{\rho\nu\mu}$.

Remark 2.3 (Convention for I_2). I_2 is defined with the *cyclic* contraction $D_{\rho\mu\nu}D^{\mu\nu\rho}$. This is the convention of the author and of Ref. [22], and it is used throughout. The other three-cycle gives nothing new, $D_{\rho\mu\nu}D^{\nu\rho\mu} = I_2$ identically, whereas the *transposed* contraction $I_5 = D_{\rho\mu\nu}D^{\rho\nu\mu}$ is a further, independent invariant. The two choices are different quadratic forms and lead to different, equally consistent truncations. Replacing I_2 by I_5 in everything below changes the following (all computed by `classical_checks.py` and `dof_transposed.py`):

- the axial value: $-6(\beta_1 + \beta_2)\tilde{a}^2 \rightarrow -6(\beta_1 - \beta_5)\tilde{a}^2$, so the positivity condition on the axial mass is $\beta_1 - \beta_5 > 0$ instead of $\beta_1 + \beta_2 > 0$, and $m_a^2 = 12(\beta_1 - \beta_5)g_a^2\phi_0^2$;
- the shift-invariant family of Section 2.5: in $\text{span}(I_1, I_5, I_3)$ it is the line generated by $I_1 - I_5$, not by $2I_2 + I_3$;
- the quadratic form on the 48 components $\hat{q}$: eigenvalues $\beta_1 - \beta_5$ on 16 and $\beta_1 + \beta_5$ on 32 components, instead of $\beta_1 + \beta_2$ on 16 and $\beta_1 - \frac{1}{2}\beta_2$ on 32; and on the vector sector $(\tilde{P}, \tilde{Q}, \tilde{R})$ the eigenvalues at $\beta_3 = 0$ are $3(\beta_1 - \beta_5)$, $3(\beta_1 + \beta_5)$, $6(\beta_1 + \beta_5)$, so the non-degeneracy conditions become $\beta_1 \neq \pm\beta_5$ and $\det N \neq 0$;
- the contact interaction of Section 2.7 is $\mathcal{L} = J_5^2/[64(\beta_1 - \beta_5)M_{\text{Pl}}^2]$;
- the degree-of-freedom count of Section 3 is unchanged (8 for non-degenerate coefficients, 7 at $\beta_5 = \beta_1$ where the axial vector becomes massless).

Nothing in the no-go statement, in the BRST and anomaly sections or in the flow of Section 6 depends on the choice.

On the pieces of Eq. (2) the invariants are (the pieces are mutually orthogonal except for P, Q, R among themselves; $\varepsilon^{\rho\mu\nu\tau}\varepsilon_{\rho\mu\nu\sigma} = -6\delta_\sigma^\tau$)

$$I_1 = -6a^2 + 4(P^2 + Q^2 + R^2) + 2(P \cdot Q + P \cdot R + Q \cdot R) + \hat{q}_{\rho\mu\nu}\hat{q}^{\rho\mu\nu}, \quad (4)$$

$$I_2 = -6a^2 + (P^2 + Q^2 + R^2) + 5(P \cdot Q + P \cdot R + Q \cdot R) + \hat{q}_{\rho\mu\nu}\hat{q}^{\mu\nu\rho}, \quad (5)$$

$$I_3 = (P + Q + 4R)_\rho(P + Q + 4R)^\rho, \quad (6)$$

$$I_5 = +6a^2 + P^2 + Q^2 + 4R^2 + 8P \cdot Q + 2P \cdot R + 2Q \cdot R + \hat{q}_{\rho\mu\nu}\hat{q}^{\rho\nu\mu}, \quad (7)$$

with $\hat{q}_{\rho\mu\nu}\hat{q}^{\mu\nu\rho} = \hat{q}^2$ on the 16 totally symmetric components and $-\frac{1}{2}\hat{q}^2$ on the other 32. The sign of the axial term of I_5 is opposite to that of I_2 because $\varepsilon_{\rho\mu\nu\sigma}$ is odd under $\mu \leftrightarrow \nu$ and even under a cyclic permutation. Table 1 lists the values on three configurations.

Table 1: Values of the invariants on pure configurations (units of v^2 , v the nonzero vector). Script `classical_checks.py`, all entries reproduced at residual $< 10^{-12}$ on a generic metric.

configuration	I_1	I_2	I_3	I_5
$D = \varepsilon a$	-6	-6	0	+6
only $P = v$	4	1	1	1
only $R = v$	4	1	16	4

Proposition 2.4 (Independence). *No nonzero combination $\alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3$ vanishes identically. I_1, I_2, I_3, I_5 are linearly independent.*

Proof. The rows of Table 1 restricted to (I_1, I_2, I_3) have determinant $\det \begin{pmatrix} -6 & -6 & 0 \\ 4 & 1 & 1 \\ 4 & 1 & 16 \end{pmatrix} = 270 \neq 0$, so a combination vanishing on the three configurations has $\alpha = 0$. (Distinct values would not suffice; linear independence of the triples is what is needed.) The statement for four invariants is the rank of their values on 30 random D , equal to 4. $\square$

2.4 Weyl transformation of the distortion [DERIVED]

With Γ held fixed, the Levi-Civita connection shifts under $g \rightarrow \Omega^2 g$ and the mixed distortion shifts inhomogeneously and has homogeneous weight 0,

$$D'^\rho{}_{\mu\nu} = D^\rho{}_{\mu\nu} - I^\rho{}_{\mu\nu}, \quad I_{\rho\mu\nu} = g_{\rho\mu}v_\nu + g_{\rho\nu}v_\mu - g_{\mu\nu}v_\rho, \quad v_\mu = \partial_\mu \omega, \quad (8)$$

while $D'_{\rho\mu\nu} = \Omega^2(D_{\rho\mu\nu} - I_{\rho\mu\nu})$ (the factor Ω^2 multiplies the shift as well). The shift has the form of Eq. (2) with $(P, Q, R) = (1, 1, -1)v$ and no axial or traceless part. The script `classical_checks.py` computes D' at a point from the transformed Levi-Civita connection of a random metric with random first derivatives and confirms Eq. (8) at 4×10^{-15} .

Let $W^\rho{}_{\mu\nu} = \delta_\mu^\rho V_\nu + \delta_\nu^\rho V_\mu - g_{\mu\nu}V^\rho$ be the distortion of the Weyl connection; it shifts by the same $-I$ under $V_\mu \rightarrow V_\mu - \partial_\mu \omega$.

Proposition 2.5. *$\tilde{D}^\rho{}_{\mu\nu} = D^\rho{}_{\mu\nu} - W^\rho{}_{\mu\nu}$ is Weyl invariant. Its axial vector equals that of D , and $\tilde{a}_\sigma = g_{\sigma\tau}\tilde{a}^\tau$ has weight 0 while $\tilde{a}^\sigma$ has weight -2 . Hence $f_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$ and $\sqrt{-g}f_{\mu\nu}f^{\mu\nu}$ are invariant, and so is $\sqrt{-g}\phi^2 \tilde{a}_\sigma \tilde{a}^\sigma$.*

Proof. $\tilde{D}' = (D - I) - (W - I) = \tilde{D}$. W has no totally antisymmetric part, so $\tilde{a} = a$; $\tilde{D}_{\rho\mu\nu} \rightarrow \Omega^2 \tilde{D}_{\rho\mu\nu}$ and $\varepsilon^{\sigma\rho\mu\nu} \rightarrow \Omega^{-4} \varepsilon^{\sigma\rho\mu\nu}$ give $\tilde{a}^\sigma \rightarrow \Omega^{-2} \tilde{a}^\sigma$; lowering gives weight 0. The weights $4 - 2 - 2 = 0$ and $4 - 4 = 0$ follow. $\square$

In terms of the decomposition, $\tilde{D}$ has $(\tilde{P}, \tilde{Q}, \tilde{R}) = (P - V, Q - V, R + V)$ with the same a and $\hat{q}$. The weights above are verified in `partI_II_audit_weyl.py` (30 checks, none failing).

2.5 No-go for bare invariants [DERIVED]

Theorem 2.6 (No-go). *Let $F[D]$ be a quadratic form in $D_{\rho\mu\nu}$ with constant coefficients built from the metric (any combination of the 11 + 6 invariants of Section 2.3), and $S_F = \int d^4x \sqrt{-g} F[D]$ with $D = \Gamma - \hat{\Gamma}(g)$. Under $g \rightarrow \Omega^2(x)g$ with Γ fixed, S_F is invariant for all g, Γ, Ω only if $F = 0$. A constant Ω already gives $S_F \rightarrow \Omega^2 S_F$.*

Proof. For constant Ω , $\hat{\Gamma}$ is unchanged, hence $D^\rho{}_{\mu\nu}$ is unchanged and $D_{\rho\mu\nu} \rightarrow \Omega^2 D_{\rho\mu\nu}$. A parity-even invariant has two $D_{\rho\mu\nu}$ and three inverse metrics: $\Omega^{2+2-6} = \Omega^{-2}$; a parity-odd one has two D 's, one $\varepsilon^{\dots} \rightarrow \Omega^{-4} \varepsilon^{\dots}$ and one inverse metric: $\Omega^{4-4-2} = \Omega^{-2}$. With $\sqrt{-g} \rightarrow \Omega^4 \sqrt{-g}$ the density scales as Ω^{+2} . Invariance then requires $\int \sqrt{-g} F[D] = 0$ for every D . Since Γ is an independent field one may take $D(x) = \chi(x)D_0$ with constant D_0 and a bump function χ , which gives $F[D_0] = 0$ for every constant tensor D_0 , i.e. $F = 0$ as a quadratic form. $\square$

Theorem 2.6 is a statement about the homogeneous weight ($\sqrt{-g}F$ has weight +2 and constant coefficients cannot compensate it) and does not depend on the form of the inhomogeneous shift. For $F = I_1, I_2, I_3, I_5$ the homogeneous scaling is confirmed on a generic metric (checks B2). The obstruction disappears if the constants are replaced by $\beta_i \phi^2$ with the dilaton ϕ of weight -1.

Proposition 2.7 (Local variation). *Let ω be infinitesimal, $v_\rho = \partial_\rho \omega$. Then*

$$\delta \int \sqrt{-g} \sum_i \alpha_i I_i = \int \sqrt{-g} \left\{ 2\omega \sum_i \alpha_i I_i - [2\alpha_1(A+B-C) + 2\alpha_2 C - 4\alpha_3 C + 2\alpha_5(A+B-C)]^\rho \partial_\rho \omega \right\}. \quad (9)$$

Proof. Homogeneous part: each I_i has two D 's (+2 ω each) and three inverse metrics (-2 ω each), and $\delta \sqrt{-g} = 4\omega \sqrt{-g}$, giving +2 $\omega \sqrt{-g} I_i$. Inhomogeneous part: $\delta D_{\rho\mu\nu} \supset -I_{\rho\mu\nu}$ and $D^{\rho\mu\nu} I_{\rho\mu\nu} = (A+B-C) \cdot v$, so $\delta_{\text{inh}} I_1 = -2(A+B-C) \cdot v$; for I_5 the symmetry of $I_{\rho\mu\nu}$ in $\mu\nu$ gives the same, $-2(A+B-C) \cdot v$. For I_2 both factors vary, $\delta_{\text{inh}} I_2 = -T_1 - T_2$ with $T_1 = I_{\rho\mu\nu} D^{\mu\nu\rho} = (B+C-A) \cdot v$ and $T_2 = D_{\rho\mu\nu} I^{\mu\nu\rho} = (C+A-B) \cdot v$, so $\delta_{\text{inh}} I_2 = -2C \cdot v$. For I_3 , $g^{\mu\nu} I_{\rho\mu\nu} = -2v_\rho$ so $\delta_{\text{inh}} C_\rho = +2v_\rho$ and $\delta_{\text{inh}} I_3 = 4C \cdot v$. $\square$

The formula, including the term in I_5 , is verified by complex-step differentiation of the transformed density on a generic metric (`classical_checks.py`, B3, residual $\leq 2 \times 10^{-15}$). Varying only one factor of I_2 would give $2(B+C-A)$ instead of $2C$ for the coefficient multiplying $\partial\omega$, and the check fails for it.

Proposition 2.8 (Shift-invariant combinations). *A quadratic form $F[D]$ is invariant under the finite shift $D \rightarrow D - I[v]$ for all v if and only if $N_F(1, 1, -1)^T = 0$, where N_F is its 3×3 matrix on (P, Q, R) (the axial and traceless sectors are unconstrained). Hence the shift-invariant parity-even forms span an 8-dimensional subspace of the 11-dimensional space (exact linear algebra, `partI_II_audit_algebra.py`). Inside $\text{span}(I_1, I_2, I_3)$ this is the line generated by*

$$I_\star = 2I_2 + I_3, \quad (10)$$

inside $\text{span}(I_1, I_2, I_3, I_5)$ the plane generated by $I_\star$ and $I_1 - I_5$, and inside $\text{span}(I_1, I_5, I_3)$ the line generated by $I_1 - I_5$. For each of them, $F[D'] = \Omega^{-2} F[D]$ pointwise for finite local Ω ; hence $\sqrt{-g} \phi^2 I_\star$ and $\sqrt{-g} \phi^2 (I_1 - I_5)$ are exactly Weyl invariant, without any Weyl vector.

Proof. The shift changes (P, Q, R) by $-(1, 1, -1)v$ and leaves a and $\hat{q}$ fixed; invariance for all x is $x^T N x = (x - tv_0)^T N (x - tv_0)$ for all t , i.e. $Nv_0 = 0$ for symmetric N , three conditions on the six parameters of N . From Eqs. (4)–(7), in the basis (P, Q, R) the matrices applied to $v_0 = (1, 1, -1)$ give $N_1 v_0 = N_5 v_0 = (4, 4, -2)$, $N_2 v_0 = (1, 1, 4)$ and $N_3 v_0 = -2u = (-2, -2, -8)$ with $N_3 = uu^T$, $u = (1, 1, 4)$; hence $(2N_2 + N_3)v_0 = 0$ and $(N_1 - N_5)v_0 = 0$ (checks C1, C2 of the script, which prints the matrices). A combination $\alpha_1 N_1 + \alpha_5 N_5 + \alpha_3 N_3$ annihilates v_0 only for $\alpha_3 = 0$, $\alpha_1 + \alpha_5 = 0$. The homogeneous weight -2 of I_* is that of any invariant. The finite statement is verified with D' computed from the transformed Levi-Civita connection at a point, two random trials, residual $\leq 10^{-13}$ (B4, B5); the same test shows that I_1, I_2, I_3, I_5 individually are not invariant. $\square$

Remark 2.9 (What the no-go does and does not force). Theorem 2.6 forces a compensator of weight -2 (here ϕ^2). It does *not* force the Weyl vector V_μ : the shift-invariant forms need none. The gauged construction of the next subsection, with the three free coefficients β_i multiplying $I_i[\tilde{D}]$ and a Weyl vector, is a choice that allows more operators, not a necessity.

2.6 The truncated Weyl-invariant action [DERIVED]

The truncation used throughout is

$$S_{UV} = \int d^4x \sqrt{-g} \left[\frac{1}{12} \phi^2 \tilde{R} - \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi - \frac{1}{4g_V^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4g_a^2} f_{\mu\nu} f^{\mu\nu} \right. \\ \left. + \phi^2 \sum_{i=1}^3 \beta_i I_i[\tilde{D}] - \lambda \phi^4 + \bar{\psi} (i\gamma^\mu \hat{\nabla}_\mu - y\phi) \psi + \frac{3}{2} \tilde{a}_\sigma \bar{\psi} \gamma^\sigma \gamma_5 \psi \right], \quad (11)$$

$F = dV$, $f = d\tilde{a}$. Each term has total weight zero: $\sqrt{-g}\phi^2 \tilde{R}$ and $\sqrt{-g}\phi^2 I_i[\tilde{D}]$ ($4 - 2 - 2$; $I_i[\tilde{D}] \rightarrow \Omega^{-2} I_i[\tilde{D}]$ because $\tilde{D}^\rho{}_{\mu\nu}$ is invariant), $\sqrt{-g}(D\phi)^2$ ($4 - 2 - 2$), the two field strengths ($4 - 4$) and $\sqrt{-g}\phi^4$ ($4 - 4$); for the fermion, ψ has weight $-\frac{3}{2}$, the inverse vierbein -1 , and $\tilde{a}_\sigma$ weight 0 . The Weyl vector does not couple to ψ (a term $-\frac{3}{2}iV_\mu \bar{\psi} \gamma^\mu \psi$ would be anti-Hermitian; the Levi-Civita Dirac action is already invariant for $\psi \rightarrow \Omega^{-3/2}\psi$, using $\gamma_c \gamma^{cb} = -3\gamma^b$). The coupling $\frac{3}{2}\tilde{a}_\sigma \bar{\psi} \gamma^\sigma \gamma_5 \psi$ comes from the spin-connection term $\omega_{\mu[ab]} \supset D_{[a|\mu|b]}$: only the axial part εa couples Hermitianly, the vector $P - R$ couples anti-Hermitianly and $\hat{q}$ drops out by its trace properties (verified with explicit gamma matrices in `partI_II_audit_fermion.py`, with $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ tied to $\varepsilon^{0123} = +1$).

What is assumed. (i) The distortion enters only through I_1, I_2, I_3 of the eleven even invariants; (ii) there are no terms linear in $\tilde{D}$ and no derivative couplings of $\tilde{D}$ other than f^2 (the kinetic term of $\tilde{a}$ is added by hand); (iii) matter is minimally coupled. None of these follows from Weyl symmetry. The action is a Weyl-invariant truncation, not a classification, and the coefficients β_i are free.

Algebraic sector. The components of $\tilde{D}$ other than $\tilde{a}$ have no kinetic term. With $\mathbf{x} = (\tilde{P}, \tilde{Q}, \tilde{R})$, Eqs. (4)–(6) give

$$\sum_i \beta_i I_i[\tilde{D}] = -6(\beta_1 + \beta_2) \tilde{a}^2 + \mathbf{x}^T N \mathbf{x} + (\beta_1 + \beta_2) \hat{q}^2|_{16} + (\beta_1 - \frac{1}{2}\beta_2) \hat{q}^2|_{32}, \quad (12)$$

$N = \beta_1 N_1 + \beta_2 N_2 + \beta_3 uu^T$, with eigenvalues $6(\beta_1 + \beta_2)$ on $(1, 1, 1)$ and $3(\beta_1 - \frac{1}{2}\beta_2)$ twice on the orthogonal plane at $\beta_3 = 0$, and $\det N = \lambda_p^2 \lambda_s + 6\beta_3 \lambda_p (2\lambda_p + \lambda_s)$ with $\lambda_s = 6(\beta_1 + \beta_2)$, $\lambda_p = 3(\beta_1 - \frac{1}{2}\beta_2)$ (verified at three parameter points). The sources of these fields vanish in Eq. (11), so their field equations have only the zero solution if and only if

$$\beta_1 + \beta_2 \neq 0, \quad 2\beta_1 \neq \beta_2, \quad \det N \neq 0. \quad (13)$$

These components are then auxiliary with zero on-shell value; their propagator is a constant (a contact term), so no ghost-freedom condition can be attributed to them. The numerical nullity of the 64×64 algebraic form is 0 for generic coefficients, 36 at $\beta_2 = 2\beta_1$ and 20 at $\beta_1 + \beta_2 = 0$ (checked in `classical_checks.py`).

2.7 Unitary gauge and the infrared theory [DERIVED]

The Weyl symmetry is used to set $\phi = \phi_0 = \text{const} > 0$ (reachable by $\Omega = \phi/\phi_0$). The Faddeev–Popov operator on the slice $\chi = \phi - \phi_0 = 0$ is multiplication by $-\phi_0$, field independent. In this gauge $\frac{1}{12}\phi_0^2\tilde{R} \supset -\frac{1}{2}\phi_0^2V^2$ (the term $\hat{\nabla} \cdot V$ is a total derivative) and $-\frac{1}{2}(D\phi_0)^2 = -\frac{1}{2}\phi_0^2V^2$, so the V^2 terms add to $-\phi_0^2V^2$. With $V = g_V\hat{V}$, $\tilde{a} = g_a\hat{a}$:

$$m_V^2 = 2g_V^2\phi_0^2, \quad m_a^2 = 12(\beta_1 + \beta_2)g_a^2\phi_0^2, \quad M_{\text{Pl}}^2 = \frac{\phi_0^2}{6} \left(\frac{\phi_0^2}{12}\hat{R} \equiv \frac{M_{\text{Pl}}^2}{2}\hat{R} \right), \quad (14)$$

that is $m_V^2 = 12g_V^2M_{\text{Pl}}^2$ and $m_a^2 = 72(\beta_1 + \beta_2)g_a^2M_{\text{Pl}}^2$. If the dilaton kinetic term of Eq. (11) is multiplied by a free coefficient κ (the symmetry does not fix it), $m_V^2 = (1 + \kappa)g_V^2\phi_0^2$. The positivity of m_a^2 requires $\beta_1 + \beta_2 > 0$. The infrared action is

$$S_{\text{IR}} = \int d^4x \sqrt{-g} \left[\frac{\phi_0^2}{12}\hat{R} - \frac{1}{4}\hat{F}^2 - \frac{1}{2}m_V^2\hat{V}^2 - \frac{1}{4}\hat{f}^2 - \frac{1}{2}m_a^2\hat{a}^2 - \lambda\phi_0^4 \right. \\ \left. + \bar{\psi}(i\gamma^\mu\hat{\nabla}_\mu - y\phi_0)\psi + \frac{3}{2}g_a\hat{a}_\sigma\bar{\psi}\gamma^\sigma\gamma_5\psi \right], \quad (15)$$

i.e. Einstein gravity coupled to two massive vector fields, the Weyl vector (which has absorbed the dilaton as its longitudinal mode) and the axial vector, plus Dirac fermions coupled to $\hat{a}$ through the axial current. In terms of Newton’s constant, $G_N = 3/(4\pi\phi_0^2)$. The fermion is not charged under $\hat{V}$. The auxiliary components have zero on-shell value at tree level, if Eq. (13) holds.

Contrast with Einstein–Cartan. At momenta $p \ll m_a$ integrate out $\tilde{a}$: $\mathcal{L} = -6(\beta_1 + \beta_2)\phi_0^2\tilde{a}^2 + \frac{3}{2}\tilde{a}_\sigma J_5^\sigma$ with $J_5^\sigma = \bar{\psi}\gamma^\sigma\gamma_5\psi$ gives $\tilde{a}_\sigma = J_{5\sigma}/[8(\beta_1 + \beta_2)\phi_0^2]$ and

$$\mathcal{L}_{\text{eff}}^{\text{QDG}} = \frac{J_5^\sigma J_{5\sigma}}{64(\beta_1 + \beta_2)M_{\text{Pl}}^2}. \quad (16)$$

In Einstein–Cartan theory with the same fermion coupling the scalar curvature of the connection contains $+6a_\sigma a^\sigma$ in terms of the axial vector (Ref. [22], checked there by `partI_II_audit_algebra.py`), so $\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} \cdot 6a^2 + \frac{3}{2}a_\sigma J_5^\sigma$, whence $a_\sigma = -J_{5\sigma}/(4M_{\text{Pl}}^2)$ and $\mathcal{L}_{\text{eff}}^{\text{EC}} = -\frac{3}{16}J_5^2/M_{\text{Pl}}^2$. The ratio is $-1/[12(\beta_1 + \beta_2)]$: the two interactions coincide only at $\beta_1 + \beta_2 = -\frac{1}{12}$, which is excluded by $m_a^2 > 0$. For every healthy axial vector the QDG contact interaction has the opposite sign to the Einstein–Cartan one, so the infrared limit is not Einstein–Cartan theory. The arithmetic is checked in `classical_checks.py` (D1–D5); the coefficient 6 of the Einstein–Cartan term is the one input taken from Ref. [22].

3 Constraint structure at quadratic order

3.1 Counting [DERIVED]

For a Lagrangian $L(q, \dot{q})$ with Hessian $W_{ij} = \partial^2 L / \partial \dot{q}^i \partial \dot{q}^j$ the Dirac–Bergmann algorithm gives primary, secondary,...constraints; with N_{conf} configuration variables, N_1 first-class and N_2 second-class constraints,

$$N_{\text{dof}} = N_{\text{conf}} - N_1 - \frac{1}{2}N_2. \quad (17)$$

The script `partIII_hamiltonian.py` implements the algorithm for quadratic Lagrangians (field theories reduced by the real Fourier-pair $a \cos kx + b \sin kx$, so that per-pair numbers are twice the per-mode ones) and is validated on known cases (Table 2). The Proca secondary constraint is $\partial_i \pi^i + m^2 V_0 \approx 0$ with $\{\pi^0, \chi\} = -m^2 \delta^{(3)}$, second class for $m \neq 0$; the algorithm reproduces this sign and rejects $\partial_i \pi^i - m^2 V_0$.

Table 2: Degrees of freedom from Eq. (17); the QDG rows are per spatial Fourier mode of the numerical algorithm (cyclic convention, $(\beta_1, \beta_2, \beta_3) = (0.7, 0.3, 0.2)$; the last two rows use I_5 in place of I_2). Scripts `partIII_hamiltonian.py`, `dof_transposed.py`.

Theory	N_{conf}	N_1	N_2	N_{dof}
Maxwell	4	2	0	2
Proca	4	0	2	3
ADM gravity	10	8	0	2
massive Fierz–Pauli, tuned $h_{\mu\nu}^2 - h^2$	10	0	10	5
Fierz–Pauli, mass term $h_{\mu\nu}^2 - 0.7h^2$	10	0	8	6
QDG bosonic, IR (unitary gauge)	78	8	124	8
QDG bosonic, UV (with dilaton)	79	10	122	8
transposed convention, IR	78	8	124	8
transposed convention, UV	79	10	122	8

Proposition 3.1 (Degrees of freedom of the bosonic quadratic sector). *Assume Eq. (13), $\beta_1 + \beta_2 > 0$ and $g_V, g_a, \phi_0 \neq 0$. Around flat space with $\phi = \phi_0$ the quadratic bosonic action of Eq. (15) propagates 2 (graviton) + 3 (V_μ) + 3 ($\tilde{a}_\mu$) = 8 degrees of freedom, with 8 first-class constraints (linearised diffeomorphisms). With the dilaton fluctuation the count is again 8, with 10 first-class constraints (diffeomorphisms and Weyl).*

Proof. At quadratic order the sectors decouple: the metric has the linearised Einstein–Hilbert action (2, Table 2); V_μ and $\tilde{a}_\mu$ are Proca fields with $m_V^2 > 0$, $m_a^2 > 0$ (3 each); the other 60 components of $\tilde{D}$ are auxiliary by Eq. (13). The Dirac algorithm applied to the complete system of 156 variables (graviton, V , 64 components of $\tilde{D}$) returns $N_{\text{conf}} = 78$, $N_1 = 8$, $N_2 = 124$ per mode, $78 - 8 - 62 = 8$; with the dilaton, $79 - 10 - 61 = 8$. $\square$

At the degenerate points the count changes as follows (cyclic convention): $\beta_2 = 2\beta_1$ keeps 8 but 36 components drop out of the action and the first-class constraints increase to 88 per pair (an undetermined tensor); $\beta_1 + \beta_2 = 0$ makes $\tilde{a}$ a massless gauge field and gives $2 + 3 + 2 = 7$. In the transposed convention the script gives 7 at $\beta_5 = \beta_1$ and 8 with 88 first-class constraints at $\beta_5 = -\beta_1$ (Section 2.3).

3.2 What the count does not show [OPEN]

The number 8 is the number of propagating modes of the *linearised bosonic theory around flat space with constant dilaton*. It is not a statement about the non-linear constraint algebra: the detuned Fierz–Pauli row of Table 2 shows that a sixth, ghost-like mode can enter at the non-linear level for a seemingly harmless quadratic action, and no non-linear Hamiltonian analysis of the metric-affine action is given here, nor one on curved backgrounds, nor one with fermions. The Proca-sector part of the unitary-gauge reduction also presupposes that the Weyl vector behaves as a Proca field at the level of the full action, which holds at quadratic order.

Tree-level unitarity. Every pole of the free propagators coupled to conserved sources has non-negative residue: for the graviton $T_{\mu\nu} T^{\mu\nu} - \frac{1}{2} T^2 \geq 0$ on conserved T and null momentum (minimum over 2000 random samples $4.8 \times 10^{-4} > 0$); for each Proca field the numerator

$\eta_{\mu\nu} + k_\mu k_\nu / m^2$ on shell is positive semidefinite of rank 3 (smallest eigenvalue -1.2×10^{-13} over 2000 random momenta), which needs $m^2 > 0$, i.e. $\beta_1 + \beta_2 > 0$ for $\tilde{a}$. The auxiliary components have no pole. This establishes the absence of ghosts and tachyons in the *free* spectrum; it does not prove unitarity of the interacting quantum theory.

A C^2 term is generated at one loop. The classical action (11) has no $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, which is why there is no massive spin-2 mode at tree level. This does not make the quantum theory ghost-free: matter loops produce $\int C^2$ with a coefficient proportional to the Weyl-squared coefficient $c > 0$ of scalars, Dirac fermions and vectors (Table 4; $c = \frac{11}{30}$ for the QDG matter at $n_f = 3$), no symmetry of Eq. (11) forbids it (C^2 is Weyl invariant), and the matter contributions all have the same sign ($c > 0$), so they cannot cancel among themselves; the gravity-loop contribution is gauge and parametrisation dependent off shell and is not computed. A term $-\frac{1}{2f_2^2}C^2$ added to $\frac{M^2}{2}R$ produces, around flat space, a second pole in the transverse-traceless propagator with residue of opposite sign to the massless one (Stelle [12]); its mass squared is of order $f_2^2 M^2$. Whether such a pole is physical in QDG, is removed by resummation, or is absent at an interacting fixed point is not decided here. Hence the statement “QDG is ghost-free because it has no C^2 term” is *not* made, and the flow of the C^2 coupling is not computed (Section 8.3).

4 BRST–BV structure

This section is classical: the BRST transformations, their nilpotency, the antifield grading, the classical master equation and the Faddeev–Popov action in a definite gauge are derived or checked by script. The quantum master equation is only *formulated*; its obstruction is the anomaly of the gauged Weyl symmetry, whose coefficients are not known (Section 5). The BRST operator s acts from the left as a graded derivation, $s(AB) = (sA)B + (-1)^{|A|}A sB$.

4.1 Gauge algebra [DERIVED]

The fundamental fields are the metric $g_{\mu\nu}$ (the vierbein e^a_μ if fermions are present), the independent connection $\Gamma^\rho_{\mu\nu}$, the dilaton ϕ , the Weyl vector V_μ and the fermion ψ . The distortion D , the invariant $\tilde{D} = D - W[V]$ and the axial vector $\tilde{a}_\mu$ are *composite* and carry no ghosts of their own. The infinitesimal symmetries are a diffeomorphism ξ^μ , a Weyl rescaling ω and, with fermions, a local Lorentz rotation θ^{ab} :

$$\delta_\xi \Phi = \mathcal{L}_\xi \Phi \quad (\Phi = g_{\mu\nu}, \phi, V_\mu), \quad \delta_\xi \Gamma^\rho_{\mu\nu} = \mathcal{L}_\xi \Gamma^\rho_{\mu\nu} + \partial_\mu \partial_\nu \xi^\rho, \quad (18)$$

$$\delta_\omega g_{\mu\nu} = 2\omega g_{\mu\nu}, \quad \delta_\omega \phi = -\omega \phi, \quad \delta_\omega V_\mu = -\partial_\mu \omega, \quad \delta_\omega \Gamma^\rho_{\mu\nu} = 0, \quad \delta_\omega \psi = -\frac{3}{2}\omega \psi. \quad (19)$$

The inhomogeneous term of $\delta_\xi \Gamma$ is fixed by requiring that $\nabla_\mu U^\rho$ be a tensor: with $\epsilon \partial_\mu \partial_\nu \xi^\rho$ only $\epsilon = +1$ works (zero covariance violations at $\epsilon = +1$, twelve at $\epsilon = 0, -1$; script `partIII_brst.py`). Likewise $D_\mu \Phi = (\partial_\mu + w V_\mu) \Phi$ is covariant for a field of weight w while $(\partial_\mu - w V_\mu) \Phi$ is not (zero versus twelve violations for $w = -1, -\frac{3}{2}, 2$). Acting on fields by substitution,

$$[\delta_\xi, \delta_\eta] \Phi = -\delta_{[\xi, \eta]} \Phi, \quad [\delta_\xi, \delta_\omega] \Phi = -\delta_{\xi^\mu \partial_\mu \omega} \Phi, \quad [\delta_\omega, \delta_{\omega'}] \Phi = 0. \quad (20)$$

The Weyl parameters are scalars on which diffeomorphisms act by $\omega \mapsto \xi^\mu \partial_\mu \omega$, so Weyl is a normal subgroup and the gauge group is the semidirect product $\text{Weyl} \ltimes \text{Diff}$ (times local Lorentz transformations, which commute with Weyl, when fermions are included). The structure functions are field independent, which is why the algebra closes off shell.

4.2 Ghosts and nilpotency [DERIVED]

With Grassmann-odd ghosts c^μ , c_W (and λ^{ab} for Lorentz) of ghost number +1,

$$\begin{aligned} s c^\mu &= c^\nu \partial_\nu c^\mu, & s c_W &= c^\nu \partial_\nu c_W, & s g_{\mu\nu} &= \mathcal{L}_c g_{\mu\nu} + 2c_W g_{\mu\nu}, \\ s \Gamma^\rho_{\mu\nu} &= \mathcal{L}_c \Gamma^\rho_{\mu\nu} + \partial_\mu \partial_\nu c^\rho, & s \phi &= c^\nu \partial_\nu \phi - c_W \phi, & s V_\mu &= c^\nu \partial_\nu V_\mu + V_\nu \partial_\mu c^\nu - \partial_\mu c_W, \end{aligned} \quad (21)$$

and, for fermions, $s e^a_\mu = c^\nu \partial_\nu e^a_\mu + e^a_\nu \partial_\mu c^\nu + c_W e^a_\mu + \lambda^a_b e^b_\mu$, $s \lambda^{ab} = c^\nu \partial_\nu \lambda^{ab} + \lambda^{ac} \lambda_c^b$, $s \psi = c^\nu \partial_\nu \psi - \frac{3}{2} c_W \psi - \frac{1}{4} \lambda_{ab} \gamma^{ab} \psi$. The relative signs of the Lorentz terms are not a convention: with $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$ they are fixed by $s^2 = 0$ (the opposite relative sign gives $s^2 \psi \neq 0$; 8×8 real Clifford representation, `partIII_brst.py`). For gauge fixing one adds antighosts and Nakanishi–Lautrup fields, $s \bar{c}_\mu = B_\mu$, $s \bar{c}_W = B_W$, $s B = 0$. These are antighosts, not ghosts of ghosts: the symmetries are irreducible.

Proposition 4.1 (Nilpotency). $s^2 = 0$ on c^μ , c_W , λ^{ab} , $g_{\mu\nu}$, e^a_μ , $\Gamma^\rho_{\mu\nu}$, ϕ , V_μ and ψ .

Proof. General argument. Let $[\delta_A, \delta_B] = f^C_{AB} \delta_C$ with field-independent f , $s\Phi = c^A \delta_A \Phi$ and $s c^C = -\frac{1}{2} f^C_{AB} c^A c^B$. Because s acts inside $\delta_A \Phi$ as $c^B \delta_B$ and c is odd, $s^2 \Phi = (s c^A) \delta_A \Phi - c^A c^B \delta_B \delta_A \Phi = (s c^C + \frac{1}{2} f^C_{AB} c^A c^B) \delta_C \Phi = 0$, and $s^2 c^C \propto f^C_{AB} f^A_{DE} c^D c^E c^B = 0$ by the Jacobi identity. This applies with the closure relations above. *Explicit cases.* $s^2 c^\mu = (c^\rho \partial_\rho c^\nu) \partial_\nu c^\mu - c^\nu \partial_\nu (c^\rho \partial_\rho c^\mu)$: the first two terms cancel by relabelling, the remainder $-c^\nu c^\rho \partial_\nu \partial_\rho c^\mu$ vanishes because $c^\nu c^\rho$ is antisymmetric (for commuting ghosts it would not). The same holds for c_W . For ϕ , $s^2 \phi$ is a sum of terms that cancel pairwise by relabelling, $c^\mu c^\nu \partial_\mu \partial_\nu \phi = 0$, $c^\mu c_W + c_W c^\mu = 0$ and $c_W c_W = 0$. For V_μ the c_W -dependent terms $c^\nu \partial_\nu \partial_\mu c_W - \partial_\nu c_W \partial_\mu c^\nu - \partial_\mu c^\nu \partial_\nu c_W - c^\nu \partial_\mu \partial_\nu c_W$ cancel, the middle pair because $\partial_\mu c^\nu$ and $\partial_\nu c_W$ anticommute. *Machine check.* For the metric, the connection (with its inhomogeneous term), the vierbein and the spinor, nilpotency is not obvious by inspection; it is verified in an exact graded algebra (Grassmann-odd ghost jets, even field jets, rational coefficients): $s^2 = 0$ on all jets of (c, c_W, ϕ, V, g) in $d = 2, 3, 4$, of $(c, c_W, \phi, V, g, \Gamma)$ in $d = 2$ (jet order 5) and $d = 3$ (order 4), and of $(c, c_W, \lambda, e, \psi)$ in $d = 3$ and, with a real 8×8 Clifford representation, $d = 4$ Lorentzian. Negative controls fail as they must: $s c^\mu = \pm 2c \cdot \partial c$ or $-c \cdot \partial c$, $s c_W = 0$ or $-c \cdot \partial c_W$, commuting ghosts, the wrong sign of $\partial \partial c$ in $s \Gamma$, the wrong relative Lorentz sign in $s \psi$. $\square$

Proposition 4.2 (Weyl cocycles). *Up to total derivatives, $s \int d^4 x c_W \sqrt{-g} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = 0$, $s \int d^4 x c_W \sqrt{-g} E_4 = 0$, and $\int d^4 x c_W \sqrt{-g} \square R = -\frac{1}{12} s \int d^4 x \sqrt{-g} R^2$ is exact.*

Proof. For a Weyl-invariant density X with $sX = \partial_\mu (c^\mu X)$ (this is $\sqrt{-g} C^2$), $s(c_W X) = (c^\mu \partial_\mu c_W) X - c_W \partial_\mu (c^\mu X) = -\partial_\mu (c_W c^\mu X)$ since $c^\mu \partial_\mu c_W = -\partial_\mu c_W c^\mu$. For $\sqrt{-g} E_4$ the variation $\delta_\omega(\sqrt{-g} E_4) = 8\sqrt{-g} G^{\mu\nu} \nabla_\mu \partial_\nu \omega$ (verified numerically, check C3 of `partI_II_audit_weyl.py`) adds $-8c_W \sqrt{-g} G^{\mu\nu} \nabla_\mu \partial_\nu c_W$, which after integration by parts and $\nabla_\mu G^{\mu\nu} = 0$ is $8\sqrt{-g} G^{\mu\nu} \partial_\mu c_W \partial_\nu c_W = 0$ ($\partial_\mu c_W \partial_\nu c_W$ is antisymmetric, $G^{\mu\nu}$ symmetric). For $\square R$, $\delta_\omega \int \sqrt{-g} R^2 = -12 \int \sqrt{-g} \omega \square R$ (script `partIII_wz.py`, agreement to 2×10^{-9}), and $\int \sqrt{-g} R^2$ is diffeomorphism invariant. $\square$

Whether $\int c_W \sqrt{-g} C^2$ and $\int c_W \sqrt{-g} E_4$ are non-trivial elements of the cohomology of s modulo d is a classification question treated for conformal anomalies in Ref. [14] and for local BRST cohomology in Ref. [15]; only closure is used here.

4.3 Antifields, the BV action and the classical master equation [DERIVED]

In the Batalin–Vilkovisky formalism [17], to every field and ghost Φ^A one associates an antifield Φ_A^* with $\epsilon(\Phi^*) = \epsilon(\Phi) + 1$, $\text{gh}(\Phi^*) = -\text{gh}(\Phi) - 1$ and global Weyl weight $w(\Phi^*) = -w(\Phi)$ (antifields are densities). Every term $\Phi_A^* s \Phi^A$ then has Grassmann parity 0, ghost number 0 and

weight 0; this is checked row by row for $g, e, \Gamma, \phi, V, \psi, c^\mu, c_W, \lambda$ and the antighosts in `partIII_brst.py`. The BV action is

$$S_{\text{BV}} = S_{\text{cl}}[\Phi] + \int d^4x \left(\Phi_A^* s\Phi^A + c_\mu^* s c^\mu + c_W^* s c_W + \lambda_{ab}^* s \lambda^{ab} + \bar{c}^{*\mu} B_\mu + \bar{c}_W^* B_W \right), \quad (22)$$

with S_{cl} the action (11) and the BV bracket $(F, G) = \frac{\delta^R F}{\delta \Phi^A} \frac{\delta^L G}{\delta \Phi_A^*} - \frac{\delta^R F}{\delta \Phi_A^*} \frac{\delta^L G}{\delta \Phi^A}$, for which $s\Phi^A = (\Phi^A, S_{\text{BV}})$.

Theorem 4.3 (Classical master equation). $(S_{\text{BV}}, S_{\text{BV}}) = 0$.

Proof. Write $S_{\text{BV}} = S_{\text{cl}} + \int \Phi_A^* R^A[\Phi]$ with $R^A = s\Phi^A$ independent of the antifields. Then $(S_{\text{BV}}, S_{\text{BV}}) = 2 \frac{\delta^R S_{\text{BV}}}{\delta \Phi^A} R^A$ has two terms. At order zero in the antifields, $\int \frac{\delta S_{\text{cl}}}{\delta \Phi^A} R^A$ is the variation of S_{cl} under the gauge transformation with parameters replaced by ghosts, and vanishes by the invariance of S_{cl} under diffeomorphisms, Weyl rescalings (the weights of Section 2.6) and Lorentz rotations. At order one, $\int \Phi_B^* \frac{\delta^R R^B}{\delta \Phi^A} R^A = \int \Phi_B^* s(s\Phi^B) = 0$ by Proposition 4.1. There are no higher terms because R^A does not depend on the antifields; it is this, and not field independence of structure constants (the transformations of g, Γ, V are field dependent), that makes the action linear in the antifields. $\square$

4.4 Gauge fixing [DERIVED]

Use the background split $g = \bar{g} + h$ and a fermion $\Psi = \int \sqrt{\bar{g}} [\bar{c}^\nu (F_\nu + \frac{\xi}{2} B_\nu) + \bar{c}_W (\phi - \phi_0)]$ with $F_\nu = \bar{\nabla}^\mu h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\nu h$. The Faddeev–Popov action is $S_{\text{cl}} + s\Psi$ (the overall sign of the ghost term is immaterial: it changes the Grassmann determinant by a field-independent factor). Integrating out $B_\nu = -F_\nu/\xi$ gives $-F_\nu F^\nu/(2\xi)$, and B_W enforces $\phi = \phi_0$, the unitary gauge of Section 2.7. At $\phi = \phi_0$, $s\phi = -\phi_0 c_W$ does not involve c^μ , so in the basis $(\bar{c}^\nu, \bar{c}_W) \times (c^\mu, c_W)$ the ghost operator is triangular, $\begin{pmatrix} \Delta_{\text{FP}}^\nu{}_\mu & 2\partial^\nu \\ 0 & \phi_0 \end{pmatrix}$, and its determinant is $\det \Delta_{\text{FP}} \times \det \phi_0$, the second factor being ultralocal and field independent. In unitary gauge the Weyl ghosts decouple and only the usual diffeomorphism ghosts propagate. A Lorenz gauge for V_μ cannot be combined with the unitary gauge on the same symmetry. In the symmetric phase an R_ξ -type gauge cancelling the V – χ mixing $2\phi_0 V^\mu \partial_\mu \chi$ and the χ – h mixing would be needed; it is not constructed here.

4.5 Quantum master equation and counting [OPEN]

The path-integral measure is BRST invariant if $W = S_{\text{BV}} + \hbar W_1 + \dots$ satisfies $\frac{1}{2}(W, W) = i\hbar \Delta W$. At order $\hbar^0$ this is Theorem 4.3; at order $\hbar$, $\sigma W_1 = i\Delta S_{\text{BV}}$, where ΔS_{BV} is the divergence of the BRST vector field. For diffeomorphisms and Lorentz rotations it is proportional to $\delta^4(0)$ and vanishes in a regularisation that preserves them. For the Weyl part the fields have non-zero weights, $\Delta S_{\text{BV}} \supset \int c_W \delta^4(0) \sum_A w_A$, and a covariant regularisation leaves a finite local remainder $\int c_W \mathcal{A}$ with $\mathcal{A}$ of dimension four. The quantum master equation can be solved if and only if this remainder is σ -exact up to a local counterterm; otherwise the gauged Weyl symmetry is anomalous. The candidates built from the metric alone are those of Proposition 4.2. Their coefficients for the gauged symmetry of QDG, coupled to dynamical V_μ , ϕ and Γ , have not been computed. The one-loop values of Section 5 answer a different question (the response of a matter determinant to a rescaling of a background metric). *This paper therefore makes no claim that QDG is anomaly free.*

The covariant mode count agrees with the Hamiltonian one: metric $10 - 2 \times 4 = 2$; Weyl sector $4 + 1 - 2 \times 1 = 3$ (in unitary gauge, the massive V_μ); axial vector $4 - 1 = 3$; total 8 at quadratic order. It presupposes that the unphysical modes form Kugo–Ojima quartets and decouple; this is checked by the Hamiltonian count at quadratic order but is not shown for the interacting BRST cohomology.

Table 3: Status of the BRST–BV statements.

Statement	Status
Transformations, closure of Weyl $\times$ Diff, $s^2 = 0$	derived; exact graded-jet script
Antifield grading, $(S_{\text{BV}}, S_{\text{BV}}) = 0$	derived
Faddeev–Popov action in unitary gauge, decoupling of Weyl ghosts	derived
Closure of the Weyl cocycles $c_W C^2$, $c_W E_4$; $c_W \square R$ exact	derived
Quantum master equation for QDG	formulated, not checked
Anomaly of the gauged Weyl symmetry	open
Kugo–Ojima decoupling beyond quadratic order	not shown

5 The Weyl anomaly

Three objects are easily conflated. (A) The *trace anomaly of a conformal field theory in a background metric*, defined for fields with a Weyl-invariant classical action (conformal scalar, Dirac fermion, Maxwell field). (B) The *logarithmic one-loop divergence*, the integrated Seeley–DeWitt coefficient $\text{tr } a_2$ of any Laplace-type operator; for a non-conformal field it contains R^2 and is not an anomaly but renormalises couplings. (C) The *anomaly of the gauged Weyl symmetry of QDG*, the obstruction to the quantum master equation of Section 4. Only (A) and (B) are computed here.

5.1 Coefficients [DERIVED]

In four dimensions $\text{Riem}^2 = 2C^2 - E_4 + \frac{1}{3}R^2$ and $\text{Ric}^2 = \frac{1}{2}(C^2 - E_4) + \frac{1}{3}R^2$, so that for a Laplace-type operator $\Delta = -(\nabla^2 + E)$, $\text{tr } a_2 = cC^2 - aE_4 + \varrho R^2 + b\square R$, with $E_4 = \text{Riem}^2 - 4\text{Ric}^2 + R^2$. Conformal operators have $\varrho = 0$. The coefficients, computed with exact fractions from the heat-kernel coefficient a_2 [10], are in Table 4 (script `partIII_anomaly.py`).

Table 4: Coefficients of $\text{tr } a_2 = cC^2 - aE_4 + \varrho R^2 + \dots$ (c in units of $1/120$, a of $1/360$). The first three rows are the standard conformal values [13]; for the others $\varrho \neq 0$ and the divergence is not a trace anomaly.

Operator	c	a	ϱ
conformal real scalar ($E = -R/6$)	1/120	1/360	0
Dirac fermion ($-\text{tr } a_2(\not{D}^2)$)	6/120	11/360	0
Maxwell field ($\Delta_1 - 2\Delta_0$)	12/120	62/360	0
minimal real scalar ($E = 0$)	1/120	1/360	1/72
Proca field, $m \rightarrow 0$ ($\Delta_1 - \Delta_0$)	13/120	63/360	1/72
QDG matter, 2 Proca + n_f Dirac: $c = \frac{26+6n_f}{120}$, $a = \frac{126+11n_f}{360}$, $\varrho = \frac{1}{36}$			
$n_f = 3$: $c = \frac{11}{30}$, $a = \frac{53}{120}$, $\varrho = \frac{1}{36}$			

The Proca row follows from the determinant of a massive vector on a fixed background, $\det(\Delta_1 + m^2)^{-1/2} \det(\Delta_0 + m^2)^{1/2}$: Maxwell plus one *minimally coupled* Stueckelberg scalar, whose non-conformality leaves $\varrho = \frac{1}{72}$. The coefficients of C^2 and E_4 do not depend on m , and mass terms and the couplings to $\tilde{a}_\mu$ and ϕ enter E only. Assigning the Maxwell coefficients $(62/360, 12/120)$ to a Proca field would therefore be wrong. The assumption that a massive vector is Maxwell plus a scalar is the one on which the Proca counting of Section 6 also rests.

The graviton has no (c, a) pair. Einstein gravity is not Weyl invariant, so it has no pair (c, a) in the sense of the trace anomaly. That $c_{\text{grav}} = 0$ “because the classical action has no C^2 term” is a non sequitur: c is fixed by the field content of the loop, not by the classical action. The only graviton number reproduced here is on-shell and gauge fixed: for the de Donder gauge on a Ricci-flat background the graviton plus vector-ghost determinant gives $\text{tr } a_2 = \frac{53}{45} \text{Riem}^2$ (script `partIII_anomaly.py`); off shell the coefficients depend on the gauge and on the parametrisation of the metric. A graviton row $(c, a) = (0, 212/360)$ is not used.

5.2 Wess–Zumino consistency and compensation [DERIVED]

Let $\delta_\sigma \Gamma_{\text{anom}} = \int \sqrt{-g} \sigma \mathcal{A}$ for $\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$. Since Weyl transformations commute, $\int \sigma_1 \delta_{\sigma_2}(\sqrt{-g} \mathcal{A})$ must be symmetric under $\sigma_1 \leftrightarrow \sigma_2$.

Proposition 5.1 (Consistency). $\sqrt{-g} C^2$ and $\sqrt{-g} E_4$ satisfy the condition.

Proof. $\sqrt{-g} C^2$ is invariant. For E_4 , $\delta_{\sigma_2}(\sqrt{-g} E_4) = 8\sqrt{-g} G^{\mu\nu} \nabla_\mu \partial_\nu \sigma_2$ and, by $\nabla_\mu G^{\mu\nu} = 0$, $\int \sqrt{-g} \sigma_1 G^{\mu\nu} \nabla_\mu \partial_\nu \sigma_2 = - \int \sqrt{-g} G^{\mu\nu} \partial_\mu \sigma_1 \partial_\nu \sigma_2$, which is symmetric. $\square$

That $\square R$ is the variation of the local functional $-\frac{1}{12} \int \sqrt{-g} R^2$ and that C^2 and E_4 are not variations of local functionals of the metric in four dimensions are taken from Ref. [14].

Compensation by a dilaton. If a field $\tau \rightarrow \tau + \sigma$ is available, the Komargodski–Schwimmer action $S_{\text{WZ}}[g, \tau]$ [16] satisfies $\delta S_{\text{WZ}} = \int \sqrt{-g} \sigma (cC^2 - aE_4)/16\pi^2$. We do not rely on the printed form: in the conventions of this paper the cocycle condition $S_{\text{WZ}}[e^{2s}g, \tau + s] - S_{\text{WZ}}[g, \tau] = S_{\text{WZ}}[e^{2s}g, s]$ is verified numerically on S^4 with residual 4×10^{-15} , with controls (flipped sign of the $G^{\mu\nu}$ term, wrong coefficient of $(\partial\tau)^4$ or of $(\partial\tau)^2 \square\tau$) failing by 0.3–67 (script `partIII_wz.py`). The naive term $\int \sqrt{-g} \tau (cC^2 - aE_4 + b\square R)$ is *not* a solution beyond first order in τ , because $\delta(\sqrt{-g} E_4) \neq 0$ locally.

Proposition 5.2 (Compensated effective action). Let $S_{\text{WZ}}[g, \tau]$ satisfy $S_{\text{WZ}}[g, 0] = 0$ and $\delta S_{\text{WZ}} = \int \sqrt{-g} \sigma \mathcal{A}$ under $g \rightarrow e^{2\sigma}g$, $\tau \rightarrow \tau + \sigma$, and let $\delta_\sigma \Gamma_{\text{anom}}[g] = \int \sqrt{-g} \sigma \mathcal{A}$. With the invariant metric $\hat{g} = e^{-2\tau}g$,

$$S_{\text{WZ}}[g, \tau] = \Gamma_{\text{anom}}[g] - \Gamma_{\text{anom}}[\hat{g}], \quad \Gamma_{\text{inv}}[g, \tau] \equiv \Gamma_{\text{anom}}[g] - S_{\text{WZ}}[g, \tau] = \Gamma_{\text{anom}}[\hat{g}], \quad (23)$$

and Γ_{inv} is invariant under the combined transformation.

Proof. $D[g, \tau] = \Gamma_{\text{anom}}[g] - \Gamma_{\text{anom}}[\hat{g}] - S_{\text{WZ}}[g, \tau]$ is invariant because $\hat{g}$ is, and $\delta \Gamma_{\text{anom}}[g] = \delta S_{\text{WZ}}$. Choosing $\sigma = -\tau$ gives $D[g, \tau] = D[\hat{g}, 0] = 0$. $\square$

In words: a compensating Wess–Zumino term amounts to *evaluating the anomalous effective action on the Weyl-invariant metric $\hat{g} = e^{-2\tau}g$* . In unitary gauge $\tau = 0$, $\hat{g} = g$ and the compensating term vanishes.

5.3 The gauged symmetry: what is and is not established [OPEN]

In QDG the Weyl symmetry is gauged, and an anomaly is an obstruction of ghost number one to the quantum master equation. (i) The cocycles $c_W C^2$ and $c_W E_4$ are BRST closed (Proposition 4.2); the cohomology $H^1(s|d)$ for the gauged Weyl symmetry coupled to dynamical gravity has not been worked out here. (ii) An F^2 term is also allowed, with a coefficient tied to the scale dependence of g_V , independent of c . (iii) Covariantisation is not a cure: any functional of $\hat{g}$, of $D_\mu \tau = \partial_\mu \tau + V_\mu$ and of Weyl-covariant curvatures is invariant under the gauged transformation and cannot compensate an anomalous variation, so the Wess–Zumino term, which contains τ undifferentiated, is not gauge invariant, as for an anomalous U(1) with a Stueckelberg field. (iv) In unitary gauge the gauge-fixed theory is the massive-vector theory of

Section 2.7 with the anomalous effective action $\Gamma_{\text{anom}}[g]$; the compensation does not constrain results computed there.

Open Problem 5.3 (Anomaly of the gauged Weyl symmetry). Determine the coefficients of the cocycles $\int c_W \mathcal{A}$ in the UV theory (11), including the graviton, dilaton, Weyl vector, axial vector and fermions, in a gauge-independent formulation; decide whether a Weyl-covariant regularisation with the regularisation scale tied to the dilaton, $\mu \propto \phi$, removes it; and if not, determine the consequence for unitarity in unitary gauge. By analogy with anomalous abelian gauge theories with Stueckelberg fields one may expect a finite cutoff of order m_V/g ; this is a conjecture and is not derived.

6 The Einstein–Hilbert flow with QDG-like matter

This section derives the flow of the Einstein–Hilbert truncation with minimally coupled matter from the Wetterich equation, reduces its fixed-point equations to a cubic, and applies it to the matter content that QDG has in unitary gauge. It follows the asymptotic-safety scenario [1] with matter as treated in Refs. [6, 7]. It is the only place in the paper where a renormalization-group statement is made, and it is a statement about the Einstein–Hilbert sector with matter, not about the Weyl-gauged operators of Section 2.6.

6.1 Setting and assumptions

Assumption 6.1 (Truncation). The effective average action is $\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} (-R + 2\bar{\Lambda}_k) + \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}} + \Gamma_k^{\text{matter}}$ in Euclidean signature on a background S^4 , with the linear split $g = \bar{g} + h$, harmonic gauge $F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\mu h$ with gauge parameter $\alpha = 1$, and the optimised (Litim [4]) cutoff profile $R^{(0)}(z) = (1 - z)\theta(1 - z)$. Matter is minimally coupled with canonical kinetic terms; the matter wave-function renormalisations are not run. We write $\tilde{G} = k^2 G_k$, $\lambda = \bar{\Lambda}_k/k^2$, $t = \ln k$, $\eta = \partial_t \ln G_k$, $u = 1 - 2\lambda$; critical exponents θ_i are minus the eigenvalues of the stability matrix, so $\text{Re } \theta_i > 0$ means UV-relevant.

Assumption 6.2 (QDG-like content). In the infrared action (15) the field content that enters the flow of G_k and $\bar{\Lambda}_k$ is the metric, two massive vectors and n_f Dirac fermions. Each massive vector is counted as a massless vector plus one minimally coupled real scalar (its Stueckelberg mode), i.e. the limit $m^2/k^2 \rightarrow 0$. This is the relevant limit for a fixed point reached as $k \rightarrow \infty$ only if the masses stay finite along the flow, which is itself an assumption (they are proportional to ϕ_0 , whose running is not computed). The Yukawa coupling y and the axial coupling g_a are not run.

Remark 6.3 (What is left out). Assumption 6.2 does not contain the dilaton’s Weyl-covariant coupling to curvature, its derivative coupling to V_μ , any non-minimal coupling of the Stueckelberg scalars, the axial coupling of $\tilde{a}_\mu$ to the fermions, the algebraic distortion sector with its free β_i , the C^2 and R^2 operators, and the running of ϕ_0 . The relation $G_N = 3/(4\pi\phi_0^2)$ of Section 2.7 is not used to identify G_k with a running ϕ_0 . Whether these ingredients modify the flow is an open problem (Section 8.3).

6.2 Inputs [INPUT]

Decompose $h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{4} \bar{g}_{\mu\nu} \varphi$ with $\hat{h}$ traceless. In the gauge of Assumption 6.1 the quadratic form on S^4 is diagonal [3] (Eqs. (4.12), (4.13) and (4.17)–(4.19) of that reference, with $C_V = -1/d$ for the ghosts; we read them in its arXiv text):

$$\hat{h} : -\bar{D}^2 - 2\bar{\Lambda}_k + C_T R, \quad \varphi : -\bar{D}^2 - 2\bar{\Lambda}_k + C_S R, \quad \text{ghosts: } -\bar{D}^2 - \frac{1}{4} R, \quad (24)$$

with $C_T = \frac{d(d-3)+4}{d(d-1)} = \frac{2}{3}$, $C_S = \frac{d-4}{d} = 0$ for $d = 4$. The overall sign of the φ operator (the conformal-factor problem) cancels in the ratio entering the Wetterich equation [2] if the cutoff carries the same factor. The heat-kernel coefficient $\frac{1}{6}R$ per component of a Bochner Laplacian is standard [10]. These are the only inputs that are not recomputed.

6.3 Trace and flow equations [DERIVED]

The ten metric components have weight $+\frac{1}{2}$ and mass $w = -2\lambda$; the four ghost components have weight -1 and mass 0 . With threshold functions $\Phi_n^p(w)$, $\tilde{\Phi}_n^p(w)$ built from $R^{(0)}$, expanding to first order in R/k^2 at fixed η gives $\partial_t \Gamma_k = \int \sqrt{g} \{k^4 a_0 + k^2 R a_1\}$ with

$$16\pi^2 a_0 = 5[2\Phi_2^1(w) - \eta\tilde{\Phi}_2^1(w)] - 8\Phi_2^1(0), \quad (25)$$

$$16\pi^2 a_1 = \frac{5}{3}\Phi_1^1(w) - \frac{5}{6}\eta\tilde{\Phi}_1^1(w) - 6\Phi_2^2(w) + 3\eta\tilde{\Phi}_2^2(w) - \frac{4}{3}\Phi_1^1(0) - 2\Phi_2^2(0). \quad (26)$$

Matching to $\partial_t \Gamma_k$ on the Einstein–Hilbert ansatz (the coefficient of R gives η , the volume term gives β_λ):

$$\eta = 16\pi\tilde{G}a_1, \quad \beta_\lambda = -(2-\eta)\lambda + 8\pi\tilde{G}a_0, \quad \beta_{\tilde{G}} = (2+\eta)\tilde{G}. \quad (27)$$

Since a_1 is linear in η , $\eta = \tilde{G}B_1/(1-\tilde{G}B_2)$. For the Litim profile $\Phi_n^p(w) = 1/[\Gamma(n+1)(1+w)^p]$, $\tilde{\Phi}_n^p(w) = 1/[\Gamma(n+2)(1+w)^p]$ and

$$B_1 = \frac{1}{3\pi} \left[\frac{5}{u} - \frac{9}{u^2} - 7 \right], \quad B_2 = -\frac{1}{12\pi} \left[\frac{5}{u} - \frac{6}{u^2} \right], \quad 8\pi\tilde{G}a_0 = \frac{\tilde{G}}{2\pi} \left[\frac{5}{u} - 4 - \frac{5\eta}{6u} \right]. \quad (28)$$

The anomalous dimension is the coefficient of R ; it cannot be read off from the volume term, which determines only $\partial_t \bar{\Lambda}_k$.

Checks. The script `eh_flow_qdg.py` assembles B_1, B_2 and the volume term from the mode table and the threshold functions and compares them with Eq. (28): agreement at 10^{-16} at $\lambda = -0.3, 0, 0.1, 0.25$ (at $\lambda = 0$, $B_1 = -11/(3\pi)$ and $B_2 = 1/(12\pi)$). For pure gravity the system has one admissible fixed point,

$$(\tilde{G}_*, \lambda_*) = (0.7073, 0.1932), \quad \theta = 1.475 \pm 3.043i, \quad \tilde{G}_* \lambda_* = 0.137. \quad (29)$$

Lauscher and Reuter report for the pure Einstein–Hilbert truncation with $\alpha = 1$ complex exponents $\theta = \theta' \pm i\theta''$ with $1.4 \lesssim \theta' \lesssim 1.8$, $2.3 \lesssim \theta'' \lesssim 4$ and $\tilde{G}_* \lambda_* \approx 0.12$ for the cutoffs they study (Section 5.3.1, item 4E.H., of Ref. [5], read in the PDF); our values are in those ranges and the product has the same size. We have not compared individual digits with a published optimised-cutoff table.

6.4 Minimally coupled matter [DERIVED]

For a field with fluctuation operator $\Delta = -\bar{D}^2 + E$ on a bundle of fibre dimension N , $\text{tr } b_2 = \frac{R}{6}N - \text{tr } E$ on S^4 . We take a cutoff that is a function of the full operator Δ (“type B”), so that E is absorbed in the argument of the threshold functions. Each field contributes $\partial_t \Gamma_k \ni \int \sqrt{g} \{k^4 Y \Phi_2^1(0)/(16\pi^2) + k^2 R X \Phi_1^1(0)/(96\pi^2)\}$, normalised so that a real minimally coupled scalar has $X = Y = 1$. Matching as in Eq. (27) gives for the Litim cutoff

$$\Delta\eta = \frac{\tilde{G}X}{6\pi}, \quad \Delta\beta_\lambda = \frac{\tilde{G}Y}{4\pi}, \quad X = N_S + 2N_D - 4N_V - 3N_P, \quad Y = N_S - 4N_D + 2N_V + 3N_P, \quad (30)$$

for N_S real scalars, N_D Dirac fermions, N_V massless vectors and N_P Proca fields in the massless limit. The coefficients are: Dirac, $\not{D}^2 = -\bar{D}^2 + \frac{R}{4}$, $N = 4$, weight $-\frac{1}{2}$, $X_D = 2$, $Y_D = -4$; massless vector in Feynman gauge, $E = R^\mu{}_\nu$ ($\text{tr } E = R$) with two scalar ghosts of weight -1 , $X_V = -4$, $Y_V = 2$; Proca in the massless limit = vector + scalar, $X_P = -3$, $Y_P = 3$.

Comparison with the literature. Donà, Eichhorn and Percacci give in Eqs. (35)–(36) of Ref. [6] (their one-loop approximation with a “type II” cutoff) $\beta_{\tilde{G}} \ni \frac{\tilde{G}^2}{6\pi}(N_S + 2N_D - 4N_V - 46)$ and $\beta_{\tilde{\lambda}} \ni \frac{\tilde{G}}{4\pi}(N_S - 4N_D + 2N_V + 2)$, which we read in the arXiv PDF. The matter combinations $N_S + 2N_D - 4N_V$ and $N_S - 4N_D + 2N_V$ and the prefactors $\frac{1}{6\pi}$ and $\frac{1}{4\pi}$ coincide with Eq. (30). That their type II is our type B was not checked, and the finite- η terms were not compared. *The value of X depends on the matter-cutoff type:* with a cutoff that is a function of $-\bar{D}^2$ alone (“type A”), E stays in the propagator, the same counting gives $X_D = -1$, $X_V = -1$, $X_P = 0$ and Y is unchanged (script `sensitivity_xy.py`). Matter enters the numerator of η but not the denominator $1 - \tilde{G}B_2$.

For n_f Dirac fermions and two Proca fields

$$X = 2n_f - 6, \quad Y = 6 - 4n_f, \quad (31)$$

and for two massless vectors $X = 2n_f - 8$, $Y = 4 - 4n_f$; the two coefficients must come from the same counting.

6.5 The system and its fixed points [DERIVED]

$$\beta_{\tilde{G}} = (2 + \eta)\tilde{G}, \quad \eta = \frac{\tilde{G}[B_1(\lambda) + X/(6\pi)]}{1 - \tilde{G}B_2(\lambda)}, \quad \beta_{\lambda} = -(2 - \eta)\lambda + \frac{\tilde{G}}{2\pi} \left[\frac{5}{u} - 4 - \frac{5\eta}{6u} + \frac{Y}{2} \right]. \quad (32)$$

A non-Gaussian fixed point is *admissible* if $\tilde{G}_* > 0$, $\lambda_* < \frac{1}{2}$ and $1 - \tilde{G}_*B_2(\lambda_*) > 0$, i.e. on the same side of the pole of η as the Gaussian fixed point; in Section 6.6 we add $\text{Re } \theta_i > 0$ where stated.

Proposition 6.4 (Closed form of the fixed points). *The admissible non-Gaussian fixed points of Eq. (32) are*

$$\lambda_* = \frac{1 - u}{2}, \quad \tilde{G}_* = \frac{2\pi}{N(u)}, \quad N(u) = \frac{4}{u^2} - \frac{5}{2u} + \frac{c}{6}, \quad c = 14 - X, \quad (33)$$

where u is a root of

$$2cu^3 - (2c + 54 - 3Y)u^2 + 118u - 48 = 0 \quad (34)$$

with $u > 0$, $N(u) > 0$ and $N(u) - 2\pi B_2(u) = \frac{3}{u^2} - \frac{5}{3u} + \frac{c}{6} > 0$.

Proof. For $\tilde{G} \neq 0$, $\beta_{\tilde{G}} = 0$ means $\eta = -2$, i.e. $\tilde{G}[B_1 + X/(6\pi) - 2B_2] = -2$, so $\tilde{G} = 2\pi/N$ with $N = -\pi B_1 + 2\pi B_2 - X/6$; with Eq. (28), $-\pi B_1 = -\frac{5}{3u} + \frac{3}{u^2} + \frac{7}{3}$ and $2\pi B_2 = -\frac{5}{6u} + \frac{1}{u^2}$, which gives $N(u)$. With $\eta = -2$, $\beta_{\lambda} = 0$ reads $4\lambda = \frac{\tilde{G}}{2\pi} [\frac{20}{3u} - 4 + \frac{Y}{2}]$. Substituting $\tilde{G}$, multiplying by $6u^2 N(u)$ and using $\lambda = (1 - u)/2$ gives $2(1 - u)(24 - 15u + (14 - X)u^2) = u(40 + (3Y - 24)u)$, which is Eq. (34). Conversely each root satisfying the inequalities is a fixed point; the last inequality is $1 - \tilde{G}_*B_2 = 1 - 2\pi B_2/N > 0$ with $N > 0$. $\square$

For the QDG-like content, $(40 - 4n_f)u^3 - (76 + 8n_f)u^2 + 118u - 48 = 0$. At $n_f = 3$ it is $28u^3 - 100u^2 + 118u - 48 = 0$ with the single real root $u = 1.6358$; there the full flow gives $\eta = -2.000000000000$ and $\beta_{\lambda} = 9 \times 10^{-16}$. As an independent check on the cubic, 2000 random starts of a Newton solver on Eq. (32) ($\tilde{G} \in [0.03, 100]$, $\lambda \in [-40, 0.45]$) converge only to this zero.

The branch is continuous in n_f . λ_* changes sign at $n_f = 17/6$ (there $u = 1$ solves the cubic), and the pair of exponents turns from complex to real at $n_f \simeq 2.873$, so $n_f = 3$ lies just beyond a qualitative change: for $n_f \leq 2$ the fixed point is the familiar Einstein–Hilbert one with $\lambda_* > 0$ and complex exponents, for $n_f \geq 3$ it has two real relevant directions. A fixed point exists for every $n_f < 10$ (scan of $n_f \in [0, 10]$ in steps of 0.01, no gap). At $c = 14 - X \rightarrow 0$, i.e. $n_f \rightarrow 10$, one root of the cubic escapes to infinity, with $\lambda_*(10 - n_f) \rightarrow -19.5$ and $\tilde{G}_*(10 - n_f) \rightarrow 52\pi/7 = 23.34$

Table 5: Admissible fixed point for n_f Dirac fermions and two Proca fields in the massless limit, $(X, Y) = (2n_f - 6, 6 - 4n_f)$ (Litim cutoff, type-B matter). Script `eh_flow_qdg.py`; $\max |\beta_i| < 10^{-12}$ in every row and $1 - \tilde{G}_* B_2$ lies in $[0.84, 1.11]$.

n_f	X	Y	$\tilde{G}_*$	λ_*	$\tilde{G}_* \lambda_*$	$\theta_{1,2}$
0	-6	6	0.5093	0.2290	0.117	$2.171 \pm 3.263 i$
1	-4	2	0.6204	0.2030	0.126	$1.760 \pm 3.023 i$
2	-2	-2	0.8362	0.1568	0.131	$1.226 \pm 2.531 i$
3	0	-6	2.7319	-0.3179	-0.868	2.207, 0.723
4	2	-10	3.9037	-1.0676	-4.17	1.797, 3.161
6	6	-18	5.9848	-2.8594	-17.1	2.146, 3.695
8	10	-26	11.836	-7.8175	-92.5	2.249, 3.899
9.5	13	-32	46.841	-37.110	-1738	2.286, 3.980

(checked at $n_f = 9.9, 9.99, 9.999$), and a scan of n_f from 10 to 200 in steps of 0.01 finds no admissible root (a numerical observation; for two massless vectors the end is at $n_f = 11$). The value 10 belongs to the type-B matter counting (Section 6.7). The qualitative statement that there is a maximal number of matter degrees of freedom compatible with a fixed point agrees with the abstract of Ref. [6].

Reliability at $n_f = 3$. The fixed point has $\tilde{G}_* = 2.73$, $\lambda_* = -0.318$, $\tilde{G}_* \lambda_* = -0.87$, whereas for pure gravity and $n_f \leq 2$ the product is ≈ 0.12 – 0.14 . We call a fixed point *within the conventional range* if $\tilde{G}_* < 2$ and $\lambda_* > -1$; these numbers are a convention for where an expansion in R/k^2 with higher operators neglected is plausible, not a derived bound. The $n_f = 3$ point is just outside it for the optimised cutoff with type-B matter, and moves quickly outside for larger n_f ($|\lambda_*| > 5$ for $n_f \gtrsim 7.3$). The truncation gives no reason to trust the numerical position of the $n_f = 3$ point.

6.6 Regulator dependence [DERIVED]

We repeated the calculation for $R^{(0)} = a(1 - z)\theta(1 - z)$ with $a = 0.5, 0.75, 1, 1.5, 2, 3$ and the exponential profile $R^{(0)} = a z/(e^z - 1)$ with $a = 0.5, 1, 2$, computing the threshold functions by Gauss–Legendre quadrature (scripts `rre.py`, `rre_run50.py`, `rre_qdg.py`). Let ρ be the fraction of the nine regulators for which an admissible fixed point with $\text{Re } \theta_i > 0$ exists (the regulator-robust-existence criterion of Ref. [22]; a necessary condition only, silent on gauge, truncation and position).

Table 6: The fixed point closest to $\lambda = 0$ at $n_f = 3$ (two Proca, $X = 0$, $Y = -6$) for nine regulators. Script `rre_qdg.py`.

profile	a	$\tilde{G}_*$	λ_*	$\theta_{1,2}$
optimised	0.5	2.7888	-0.1206	1.692, 0.280
optimised	0.75	2.8339	-0.2426	2.089, 0.555
optimised	1	2.7319	-0.3179	2.207, 0.723
optimised	1.5	2.5703	-0.4266	2.310, 0.926
optimised	2	2.4659	-0.5090	2.359, 1.053
optimised	3	2.3434	-0.6354	1.214, 2.410
exponential	0.5	0.8067	0.0773	$0.091 \pm 1.856 i$
exponential	1	0.6512	0.0578	$0.407 \pm 1.391 i$
exponential	2	0.5804	-0.0714	$0.600 \pm 0.718 i$

Existence is robust. For two Proca fields $\rho = 1$ for every $n_f \leq 9$ (grid step 0.5), $\rho = 0.78$ at 9.5, 0.22 at 10 and 0 for $n_f \geq 10.5$; for two massless vectors $\rho = 1$ up to $n_f = 10$, 0.78 at 10.5, 0.22 at 11. At $n_f = 3$ all nine regulators have an admissible fixed point. *Position and character are not robust.* At $n_f = 3$, $\tilde{G}_*$ ranges from 0.58 to 2.83; for the six optimised regulators $\tilde{G}_* > 2.3$ with real exponents and $\lambda_* \in [-0.64, -0.12]$, for the three exponential regulators $\tilde{G}_* < 0.81$ with a complex pair of small real part. The conventional range ($\tilde{G}_* < 2$, $\lambda_* > -1$) holds for three of nine regulators, $\lambda_* > -1$ for all nine. The n_f at which the exponents become real is 2.8–3.0 for the optimised and 3.1–3.2 for the exponential regulators, so $n_f = 3$ is on different sides of the change of character depending on the regulator. *The farther the larger:* for $n_f = 4, 6, 8$ the maximum of $\tilde{G}_*$ over the nine regulators is 4.95, 7.54, 13.9 and the minimum of λ_* is -3.0 , -8.3 , -21.2 . Lauscher and Reuter note that existence is a scheme-independent feature of the exact theory whereas the precise location, and in a truncation the exponents, are not [5]; the nine-regulator result is the first statement applied to a small family, not a proof for the full theory.

6.7 Matter-cutoff type and sensitivity to the omitted sector [DERIVED]

The flow depends on the matter content only through (X, Y) . The ingredients of Remark 6.3 change X and Y by amounts that cannot be computed here; we ask instead how far (X, Y) can move from $(0, -6)$ before the $n_f = 3$ point is lost or changes character (script `sensitivity_xy.py`, Litim $a = 1$, exact cubic). This is a sensitivity analysis, not a computation of the omitted sector.

Table 7: The fixed point of Eq. (32) for $(X, Y) = (0, -6) + (\delta X, \delta Y)$. A scalar with non-minimal coupling ξR has $X = 1 - 6\xi$ in the type-B counting (same Y), so $\delta X = -1$ is $\xi = \frac{1}{6}$ for one scalar.

δX	δY	$\tilde{G}_*$	λ_*	$\theta_{1,2}$
0	0	2.7319	-0.3179	2.207, 0.723
-1	0	2.3341	-0.2204	1.997, 0.629
-3	0	1.8188	-0.1132	1.491, 0.978
+2	0	3.6833	-0.5688	1.131, 2.514
0	+1	1.3892	0.0518	$0.656 \pm 1.363 i$
0	-1	3.0159	-0.5000	1.206, 2.579
0	+2	1.0868	0.1130	$0.820 \pm 2.055 i$
0	-2	3.1232	-0.6264	1.437, 2.791
+2	+2	1.0875	0.1250	$0.697 \pm 2.228 i$

Existence is stable: along $Y = -6$ an admissible fixed point exists for every $X \in [-100, 277.35]$ and along $X = 0$ for every $Y \in [-100, 100]$ (windows searched). *Position and character are not:* a shift of +1 in Y moves the point from $(2.73, -0.32)$ to $(1.39, 0.05)$ and turns the exponents into a complex pair; a shift of -3 in X moves it inside the conventional range. *The matter-cutoff type is such a shift.* With type A ($X_D = X_V = -1$, $X_P = 0$, so $X = -n_f$ for the QDG-like content, Y unchanged) the $n_f = 3$ point is $(\tilde{G}_*, \lambda_*) = (1.819, -0.113)$ with real exponents (1.491, 0.978), inside the conventional range; for $n_f = 4, 6, 10, 20, 40$ it stays at $\tilde{G}_* = 2.30, 2.13, 1.74, 1.18, 0.72$ and $\lambda_* = -0.53, -0.90, -1.33, -1.86, -2.30$, so there is no end of the branch at $n_f = 10$. The type-B counting reproduces the combinations of Ref. [6] and is used throughout; the point is that “ $n_{\text{crit}} = 10$ ” belongs to it.

Remark 6.5 (Standard Model content). For $N_S = 4$, $N_D = 22.5$, $N_V = 12$ (minimal coupling only; $X = 1$, $Y = -62$) the same equations give $(\tilde{G}_*, \lambda_*) = (3.217, -4.393)$ with $\theta = (2.083, 3.926)$; adding the two QDG vectors as Proca fields ($X = -5$, $Y = -56$) gives

(2.187, -2.693) with $\theta = (2.044, 3.883)$ and an admissible fixed point for all nine regulators (script `sm_plus_qdg.py`). These are applications of Eq. (30), not QDG predictions; the identification of the QDG fermions with the Standard-Model ones is not made.

6.8 Meaning for QDG [OPEN]

In this truncation (optimised cutoff, harmonic gauge $\alpha = 1$, linear split, S^4 , type-B minimally coupled matter) the Einstein–Hilbert flow with QDG-like matter has a non-Gaussian fixed point for every $n_f < 10$, regulator-robust in existence up to $n_f \simeq 9$, with real exponents at $n_f = 3$. This is an existence statement in a truncation. It is not a statement of asymptotic safety of QDG, for three reasons. (i) The Weyl-gauged sector (Remark 6.3) is not in the flow. (ii) Gauge and parametrisation dependence were not studied: λ_* and $\tilde{G}_*$ are not gauge invariant, so the numbers are statements about $\alpha = 1$ and the linear split. (iii) The position of the $n_f = 3$ point varies by a factor of five with the regulator and moves into or out of the conventional range with the matter-cutoff type. The gravitational contribution to the running of the gauge couplings is known to depend on the gauge [9]; for that reason no beta functions for g_V , g_a or y are given.

7 The phenomenological model flow [MODEL]

Earlier versions used the system

$$\eta = -\frac{\tilde{G}A(\lambda)}{1 - \tilde{G}B(\lambda)} + C_\eta \tilde{G}, \quad \beta_{\tilde{G}} = (2 + \eta)\tilde{G}, \quad (35)$$

$$\beta_\lambda = -(2 - \eta)\lambda + \frac{\tilde{G}}{2\pi}B(\lambda) + C_{\text{bl}}\tilde{G} + \frac{72\lambda^2}{16\pi^2}, \quad (36)$$

with $A = \frac{5}{1-2\lambda} - \frac{3}{1-\lambda} - 1$, $B = A + 1$, $C_\eta = \frac{n_f-4}{3\pi}$, $C_{\text{bl}} = \frac{3-2n_f}{2\pi}$. *This is a model, not a derivation from QDG or from the Einstein–Hilbert action* (script `model_flow_check.py`):

1. A and B have the form of signed multiplicities of the *volume* term ($5 = \frac{9}{2} + \frac{1}{2}$ metric components at mass -2λ ; $-\frac{3}{1-\lambda} - 1$ standing for ghost-like contributions) but are used as the coefficient of R , which in the derived flow is given by B_1, B_2 of Eq. (28). They are not a reproduction of any published function that we have checked.
2. The model has poles at $\lambda = \frac{1}{2}$ and $\lambda = 1$; the derived flow only at $\lambda = \frac{1}{2}$. No rescaling of $\tilde{G}$ or λ maps one onto the other, and the model is not a gauge or parametrisation variant of the derived flow in the harmonic gauge.
3. The term $72\lambda^2/(16\pi^2)$ is not derived from any action.
4. C_η corresponds to two massless vectors ($X = 2n_f - 8$) whereas $C_{\text{bl}} = Y/(4\pi)$ with $Y = 6 - 4n_f$ corresponds to two Proca fields; a derived flow would use the same counting for both.
5. Its pure-gravity limit ($C_\eta = C_{\text{bl}} = 0$) is $(\tilde{G}_*, \lambda_*) = (0.3502, 0.0313)$, not the Einstein–Hilbert fixed point $(0.7073, 0.1932)$.

Whether the model can be obtained from another gauge or from the distortion sector is not known. For the record, its numbers are internally correct: it has a fixed point at $n_f = 3$, $(\tilde{G}_*, \lambda_*) = (0.4318, -0.0194)$, $\theta = (9.446, 3.519)$, together with a second solution on the other side of the pole of η at $(5.149, -0.481)$ and a large-coupling one at $(362.5, -15.66)$; Table 8 gives other n_f . The physical root is lost when $1 - \tilde{G}_*B(\lambda_*) \rightarrow 0$, at $(\tilde{G}, \lambda) = (1, (2 - \sqrt{6})/2)$ where $A = 0$, $B = 1$, which defines $n_{\text{crit}}^{\text{model}} = 4.896577961599\dots$ (closed form in the script; bisection on the numerical branch gives 4.896571).

Table 8: Fixed points of the model flow, Eqs. (35)–(36) (physical branch: $\tilde{G}_* > 0$, $\lambda_* < \frac{1}{2}$, $1 - \tilde{G}_* B > 0$, smallest $|\lambda_*|$), compared with the derived flow of Section 6. Scripts `model_flow_check.py`, `eh_flow_qdg.py`.

n_f	model [MODEL]		derived flow [DERIVED]	
	$(\tilde{G}_*, \lambda_*)$	$\theta_{1,2}$	$(\tilde{G}_*, \lambda_*)$	$\theta_{1,2}$
0	(0.2964, 0.0659)	9.86, 3.65	(0.5093, 0.2290)	$2.17 \pm 3.26 i$
1	(0.3278, 0.0438)	9.71, 3.62	(0.6204, 0.2030)	$1.76 \pm 3.02 i$
2	(0.3699, 0.0165)	9.57, 3.59	(0.8362, 0.1568)	$1.23 \pm 2.53 i$
3	(0.4318, -0.0194)	9.45, 3.52	(2.7319, -0.3179)	2.21, 0.72
4	(0.5411, -0.0733)	9.45, 3.37	(3.9037, -1.0676)	1.80, 3.16

The model keeps no claim of this paper: its fixed point is not evidence for or against the asymptotic safety of QDG. It is retained only so that numbers quoted from earlier versions can be located and read for what they are.

8 Discussion

8.1 Status of the results

Table 9 lists each result with the status labels used in the paper.

8.2 Claims of earlier versions that are not repeated

The following statements of earlier versions [18, 19] are withdrawn or changed. (i) Two phenomenological flows appeared: a polynomial one whose coefficients were set by power counting (fixed point (0.50, 0.43) with four relevant directions in a six-coupling model), which is dropped, and a later one-loop analysis whose fixed point $(\tilde{G}_*, \lambda_*) \approx (0.43, -0.02)$ was presented as derived from first principles, although its functions A, B are not derived from the Einstein–Hilbert action; it is kept, as the model of Section 7, and no claim rests on it. (ii) The local variation formula of the no-go statement was given in two mutually inconsistent forms and its proof varied only one factor of I_2 ; Proposition 2.7 replaces it, and the conclusion of the no-go (which rests on the homogeneous weight) is unchanged. (iii) The degree-of-freedom counting was not based on the Dirac algorithm and a wrong sign of the secondary Proca constraint was used; the count of Section 3 replaces it, and “no spin-2 ghost” is restricted to the tree-level free spectrum. (iv) The Wess–Zumino term $\int \tau \mathcal{A}$ is not a solution of the compensation condition beyond first order in τ ; a graviton row (c, a) and the assignment of Maxwell coefficients to Proca fields are withdrawn (Section 5). (v) A matter coefficient $a_V = +\frac{3}{2}$, a “reproduction” of published functions A, B , a statement that the number of relevant directions is gauge independent, and a heat-kernel mixing argument for the graviton–Proca sector are not derivable and are dropped. (vi) The applications (black holes, entanglement entropy, cosmology, inflation, gravitational waves, spectral dimension, swampland criteria) are not part of this paper and nothing is claimed about them.

8.3 Open problems

Open Problem 8.1 (Weyl-gauged sector in the flow). Compute the contribution to η and β_λ of the dilaton, the Weyl vector, the axial vector with their non-minimal couplings, masses and the algebraic distortion sector, instead of the minimally coupled massless Proca counting. Table 7 gives the size of the shifts in (X, Y) that would matter.

Table 9: Status of the results of this paper.

Result	Status	Comment
Decomposition $64 = 4 + 12 + 48$; 11 even + 6 odd invariants; independence	Derived	scripts; I_2 cyclic by convention, effect of the transposed one listed
No-go for bare invariants; local variation; shift-invariant family	Derived	homogeneous weight +2; complex-step check
Truncated Weyl-invariant action, unitary gauge, masses, $M_{\text{Pl}}^2 = \phi_0^2/6$	Derived	truncation with free β_i ; not a classification
Contact interaction differs in sign from Einstein–Cartan	Derived	EC coefficient taken from the book
8 bosonic d.o.f. at quadratic order	Derived	linearised, bosonic, flat; no non-linear count
Tree-level unitarity of the free spectrum	Derived	not unitarity of the interacting theory
Ghost-freedom	NOT CLAIMED	C^2 generated at one loop
$s^2 = 0$, classical master equation	Derived	exact graded jets
Quantum master equation; anomaly of the gauged Weyl symmetry	OPEN	coefficients unknown
Anomaly coefficients of matter; compensation = evaluation on $\hat{g}$	Derived	fixed background only
EH Hessian on S^4 , harmonic gauge	Input	Ref. [3], not rederived
Flow, B_1, B_2 , cubic (Prop. 6.4)	Derived	Litim, $\alpha = 1$, linear split, S^4
(X, Y) for matter	Derived	type B; X depends on the type
Fixed point at $n_f = 3$: $(2.73, -0.318)$, $\theta = (2.21, 0.72)$; exists for $n_f < 10$; nine regulators	Derived (numerical)	existence in a truncation
Position, exponents, reliability at $n_f = 3$	NOT ESTABLISHED	regulator and matter-cutoff dependent
Model flow	Model	not derived; no claim rests on it
Asymptotic safety of QDG	OPEN	Weyl-gauged sector not in the flow

Open Problem 8.2 (Truncation). Include higher-curvature operators, in particular R^2 and the Weyl-squared term whose generation at one loop was noted in Section 3.2, and the flow of the C^2 coupling. The R^2 truncation of Ref. [5] exists for pure gravity; it has not been done for the QDG-like content.

Open Problem 8.3 (Gauge, parametrisation and matter cutoff). Determine the dependence on α (including $\alpha \rightarrow 0$) and on an exponential metric parametrisation; fix the matter-cutoff convention by a principle rather than by agreement with a reference, and quantify the type-A/type-B difference of Section 6.7; test the Proca = vector + scalar identification beyond $m \rightarrow 0$. A first study of asymptotically safe generalised Proca theories with a different aim and different couplings is Ref. [8].

Open Problem 8.4 (Non-linear constraint analysis). Determine whether the non-linear metric-affine action propagates additional (Boulware–Deser-type) modes, on curved backgrounds and with fermions.

Open Problem 8.5 (Quantum master equation). Solve or obstruct the quantum master equation for Weyl $\times$ Diff coupled to V_μ , ϕ and Γ (Open Problem 5.3), and decide whether the Kugo–Ojima mechanism survives beyond quadratic order.

9 Conclusions

The classical part of Quantum Distortion Gravity is a coherent, checkable construction: a decomposition of the distortion with explicit invariants, a no-go statement that fixes the weight of the compensator and shows that the Weyl vector is a choice, a Weyl-invariant truncated action with free coefficients whose unitary-gauge limit is Einstein gravity plus two massive vectors with eight bosonic degrees of freedom at quadratic order, and a BRST structure that is nilpotent with a classical master equation. The quantum part is much less complete: the quantum master equation and the anomaly of the gauged Weyl symmetry are open, a C^2 term is generated at one loop, and the only renormalization-group result is the existence, for every $n_f < 10$, of a non-Gaussian fixed point in the Einstein–Hilbert flow with QDG-like matter, at $(\tilde{G}_*, \lambda_*) = (2.73, -0.318)$ with real exponents $(2.21, 0.72)$ for $n_f = 3$ and the optimised cutoff. Its position and the character of its exponents depend on the regulator and on the matter-cutoff convention, and the Weyl-gauged operators are not in the flow. The paper proposes QDG as a framework with a consistent classical sector and a list of precisely stated open problems; it does not establish that QDG is a consistent or asymptotically safe quantum theory.

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Institutional Review Board Statement

Not applicable. This work is a theoretical study in mathematical physics and does not involve human participants, animals, or biological samples requiring ethical approval.

Informed Consent Statement

Not applicable.

Conflicts of Interest

The author of this manuscript, **Luca Eliseo Pavesi**, is the founder, owner, and Editor-in-Chief of the *Journal of Advanced Gravitational Physics* (JAGP), ISSN **3103-8824**, the journal in which this article is published. This position constitutes a **structural conflict of interest** with respect to the editorial handling of this manuscript.

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Additional Information

This revision was prepared with the assistance of an AI language model (Claude, Anthropic), used to re-derive and cross-check the results, to write and run the verification scripts listed in Appendix A, and to edit the text. The scientific claims, the choice of conventions and the conclusions are the responsibility of the author.

Data Availability Statement

No datasets were generated or analysed. All numbers computed in this paper are reproduced by the Python scripts (NumPy and SciPy only) listed in Appendix A, distributed with the manuscript together with their output files.

A Scripts and core code

Every number computed in this paper is reproduced by one of the following scripts, stored with their output (*.out) in the `code/` directory.

- `classical_checks.py`: decomposition, expansions of I_1, I_2, I_3, I_5 , determinant 270, local Weyl transformation at a point, variation formula by complex step, finite invariance of I_* and $I_1 - I_5$, cyclic versus transposed convention, nullities, masses and contact interaction (32 checks, none failing).
- `dof_transposed.py`: Dirac–Bergmann count with I_5 in place of I_2 .
- `book_scripts/`: scripts of Ref. [22] used unmodified: `partI_II_audit_weyl.py`, `partI_II_audit_algebra.py`, `partI_II_audit_geometry.py`, `partI_II_audit_fermion.py`, `partI_II_decomposition.py`, `partI_II_nogo.py` (Part I–II); `partIII_hamiltonian.py`, `partIII_brst.py` (with `jets.py`), `partIII_anomaly.py`, `partIII_wz.py` (Part III); `eh_flow_trace.py`, `fixed_point_cubic.py`, `model_flow.py`, `rre.py`, `rre_run50.py`, `partIV_solutions_check.py` (Part IV).
- `eh_scripts/`: `eh_flow_qdg.py` (B_1, B_2 from the mode table, flow, cubic, Table 5, end of the branch, pure gravity, multi-start check); `rre_qdg.py` with `rre.py`, `rre_run50.py` (nine regulators, Table 6, $\rho(n_f)$); `sensitivity_xy.py` and `sm_plus_qdg.py` (Table 7, type-A counting, Standard-Model content); `model_flow_check.py` (Section 7).

`run_all.sh` runs them in order. The core of the derived flow and of the fixed-point search is:

```
import numpy as np
pi = np.pi

def B1B2(lam):
    # B1, B2 of Eq. (28), Litim
    u = 1 - 2*lam
    return (5/u - 9/u**2 - 7)/(3*pi), -(5/u - 6/u**2)/(12*pi)

def eta(G, lam, X):
    B1, B2 = B1B2(lam)
    return G*(B1 + X/(6*pi))/(1 - G*B2)

def beta_G(G, lam, X, Y):
    return (2 + eta(G, lam, X))*G

def beta_lam(G, lam, X, Y):
    u = 1 - 2*lam; e = eta(G, lam, X)
    return -(2 - e)*lam + G/(2*pi)*(5/u - 4 - 5*e/(6*u) + Y/2)

def fixed_points(X, Y):
    # cubic of Prop. 6.4 in u
    c = 14 - X
    roots = np.roots([2*c, -(2*c + 54 - 3*Y), 118., -48.])
    out = []
    for u in roots:
        if abs(u.imag) < 1e-9 and u.real > 0:
            u = u.real; N = 4/u**2 - 2.5/u + c/6
            if N > 0:
                G = 2*pi/N; lam = (1 - u)/2
                if 1 - G*B1B2(lam)[1] > 0: out.append((G, lam))
    return out

nf = 3
print(fixed_points(2*nf - 6, 6 - 4*nf)) # [(2.7319, -0.3179)]
```

References

- [1] S. Weinberg, Ultraviolet Divergences in Quantum Theories of Gravitation, in *General Relativity: An Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel, Cambridge University Press (1979) 790–831.
- [2] C. Wetterich, Exact Evolution Equation for the Effective Potential, *Physics Letters B* **301** (1993) 90–94.
- [3] M. Reuter, Nonperturbative Evolution Equation for Quantum Gravity, *Physical Review D* **57** (1998) 971–985, arXiv:hep-th/9605030.
- [4] D. F. Litim, Optimised renormalisation group flows, *Physical Review D* **64** (2001) 105007, arXiv:hep-th/0103195.
- [5] O. Lauscher, M. Reuter, Flow equation of quantum Einstein gravity in a higher-derivative truncation, *Physical Review D* **66** (2002) 025026, arXiv:hep-th/0205062.
- [6] P. Donà, A. Eichhorn, R. Percacci, Matter matters in asymptotically safe quantum gravity, *Physical Review D* **89** (2014) 084035, arXiv:1311.2898.
- [7] A. Eichhorn, M. Schiffer, Asymptotic safety of gravity with matter, book chapter, arXiv:2212.07456 [hep-th] (2022).
- [8] C. Pastor-Marcos, L. Heisenberg, Á. Pastor-Gutiérrez, J. M. Pawłowski, M. Reichert, Cosmology from asymptotically safe Proca theories, arXiv:2604.01090 [gr-qc] (2026).
- [9] A. R. Pietrykowski, Gauge dependence of gravitational correction to running of gauge couplings, *Physical Review Letters* **98** (2007) 061801, arXiv:hep-th/0606208.
- [10] D. V. Vassilevich, Heat Kernel Expansion: User’s Manual, *Physics Reports* **388** (2003) 279–360.
- [11] F. W. Hehl, J. D. McCrea, E. W. Mielke, Y. Ne’eman, Metric-Affine Gauge Theory of Gravity: Field Equations, Noether Identities, World Spinors, and Breaking of Dilation Invariance, *Physics Reports* **258** (1995) 1–171.
- [12] K. S. Stelle, Renormalization of Higher-Derivative Quantum Gravity, *Physical Review D* **16** (1977) 953–969.
- [13] M. J. Duff, Twenty Years of the Weyl Anomaly, *Classical and Quantum Gravity* **11** (1994) 1387–1404.
- [14] S. Deser, A. Schwimmer, Geometric Classification of Conformal Anomalies in Arbitrary Dimensions, *Physics Letters B* **309** (1993) 279–284, arXiv:hep-th/9302047.
- [15] G. Barnich, F. Brandt, M. Henneaux, Local BRST cohomology in gauge theories, *Physics Reports* (2000), DOI 10.1016/S0370-1573(00)00049-1, arXiv:hep-th/0002245.
- [16] Z. Komargodski, A. Schwimmer, On Renormalization Group Flows in Four Dimensions, *Journal of High Energy Physics* **12** (2011) 099, arXiv:1107.3987 [hep-th].
- [17] I. A. Batalin, G. A. Vilkovisky, Gauge Algebra and Quantization, *Physics Letters B* **102** (1981) 27–31.
- [18] L. E. Pavesi, On the Quantum Distortion Gravity, *Journal of Advanced Gravitational Physics* **1**, 2 (2026) (earlier version of the present paper).

- [19] L. E. Pavesi, Quantum Distortion Gravity: A Complete One-Loop Analysis of the Minimal Truncation, *Journal of Advanced Gravitational Physics* **1**, 3 (2026) (earlier version).
- [20] L. E. Pavesi, Quantum Distortion Gravity: the Einstein–Hilbert flow with QDG-like matter, companion paper, *Journal of Advanced Gravitational Physics* (2026).
- [21] L. E. Pavesi, Quantum Distortion Gravity: hidden matter sectors and the fixed point of the Einstein–Hilbert flow, companion paper, *Journal of Advanced Gravitational Physics* (2026).
- [22] L. E. Pavesi, Quantum Distortion Gravity, second edition (2026): Parts I–IV, with the scripts quoted in Appendix A.