

Conditional Bounds on the Matter Content from the Existence of an Ultraviolet Fixed Point in Quantum Distortion Gravity: Model Flow versus Derived Flow

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Abstract

The first version of this paper stated that the ultraviolet fixed point of Quantum Distortion Gravity (QDG) exists only for $n_f \leq 4.896577961599$ Dirac fermions, from a degree-7 polynomial obtained by eliminating the Newton coupling from the fixed-point equations of a one-loop system. We re-examine this claim. The polynomial and the number are reproduced exactly (all eight coefficients, with exact arithmetic, and the critical value to 10^{-13}), and the critical value is given a closed form: the admissible root leaves through the isolated singular point $(\tilde{G}, \lambda) = (1, 1 - \sqrt{6}/2)$ of the flow, where the numerator and the denominator of the anomalous dimension vanish together. This also fixes what the number is. The one-loop system is a phenomenological model, not the Einstein–Hilbert (EH) flow, and its bound is a bound on a single matter coefficient, that of the volume term, $Y \geq -13.586$, independent of the coefficient of the curvature term. In the derived EH flow (optimised cutoff, harmonic gauge, S^4 , minimally coupled matter) the corresponding bound is on the other coefficient, $X < 14$, which gives $n_f < 10$ for n_f Dirac fermions and two Proca fields (11 for two massless vectors). We give the existence region in the (X, Y) plane in closed form, show that nine regulators place the end of the branch between 9.4 and 10.8, and show that with a matter cutoff that is a function of $-\bar{D}^2$ only the bound disappears (no end up to $n_f = 10^4$). The two flows therefore give different numbers, 4.8966 and 10, for the same content, and both are conditional on the truncation, on the matter-cutoff convention and on the counting of the two QDG vectors. Neither is a bound on QDG: the Weyl-gauged sector is in neither flow.

Keywords: quantum gravity, metric-affine gravity, Weyl symmetry, asymptotic safety, functional renormalization group, fixed point, matter content, resultants

1 Introduction

Quantum Distortion Gravity (QDG) [7] is a metric-affine theory with gauged Weyl symmetry. In unitary gauge its infrared limit is Einstein gravity coupled to two massive vectors (Proca fields) and matter, with the matter content treated in this paper as n_f Dirac fermions. Whether such a theory has an ultraviolet fixed point of the functional renormalization group in the asymptotic-safety sense [1, 2, 3], and for which matter content, is the question behind a series of papers.

The first version of the present paper [9] eliminated the dimensionless Newton coupling $\tilde{G}$ from the two fixed-point equations of a one-loop system by resultants and obtained a polynomial $P(\lambda; n_f)$ of degree 7 in the dimensionless cosmological constant λ . It concluded that the physical

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fixed point exists if and only if $n_f \leq n_{\text{crit}} = 4.896577961599$, and that the Standard Model “is safely below the bound”. The title, the abstract and the sections after the polynomial presented this as an upper bound on the matter content of QDG.

Re-examination, in the light of the companion paper [8] and of the second edition of Ref. [10], leads to the following.

1. The algebra is correct and is kept. All eight coefficients of P and the number 4.896577961599 are reproduced here (Section 3); the number now has a closed form and a stated mechanism.
2. The system it refers to is not the Einstein–Hilbert flow. Its gravitational functions are mode counts of the *volume* term used as the coefficient of the *curvature* term; in the pure-gravity limit it does not reproduce the EH fixed point [8, 10]. It is a *model*. The number 4.8966 is a property of that model.
3. Within the model, the bound constrains one matter coefficient only (the volume term, Y); the corresponding bound in the derived EH flow constrains the other one (the curvature term, X), and gives 10 instead of 4.8966 for the same content (Section 4).
4. The derived bound depends on the matter-cutoff convention and, at the level of the position of the end of the branch, on the regulator; with the other standard convention it does not exist at all.

The paper is accordingly reframed as a comparison of the bounds on matter content that follow from the existence of the fixed point in the two flows, with an honest account of what each of them does and does not mean. Statements that the first version made and that do not hold are listed in Table 8.

Logical status. *Proposition* means proven in this paper given the stated definitions; *Assumption* denotes an input; *Remark* and *Open Problem* mean not proven. Headings and tables carry the tags [DERIVED] (derived here and reproduced by a script), [INPUT] (taken from a checked source), [MODEL] (phenomenological), [NUMERICAL] (established by a scan, not proven) and [OPEN].

2 The two flows and the definition of the bound

2.1 Derived Einstein–Hilbert flow with minimal matter [INPUT]

The derived flow is that of the companion paper [8]: Einstein–Hilbert truncation on a background S^4 , linear split, harmonic gauge with $\alpha = 1$, optimised (Litim) cutoff [4], Hessian constants from Ref. [3], minimally coupled matter with canonical kinetic terms. We recall the result and do not rederive it. With $\tilde{G} = k^2 G_k$, $\lambda = \bar{\Lambda}_k/k^2$, $u = 1 - 2\lambda$ and $\eta = \partial_t \ln G_k$,

$$\beta_{\tilde{G}} = (2 + \eta)\tilde{G}, \quad \eta = \frac{\tilde{G}[B_1(\lambda) + X/(6\pi)]}{1 - \tilde{G}B_2(\lambda)}, \quad \beta_\lambda = -(2 - \eta)\lambda + \frac{\tilde{G}}{2\pi} \left[\frac{5}{u} - 4 - \frac{5\eta}{6u} + \frac{Y}{2} \right], \quad (1)$$

$$B_1 = \frac{1}{3\pi} \left[\frac{5}{u} - \frac{9}{u^2} - 7 \right], \quad B_2 = -\frac{1}{12\pi} \left[\frac{5}{u} - \frac{6}{u^2} \right]. \quad (2)$$

Matter enters only through two integers: X , the coefficient of the curvature term ($\Delta\eta = \tilde{G}X/6\pi$), and Y , the coefficient of the volume term ($\Delta\beta_\lambda = \tilde{G}Y/4\pi$). They are normalised so that a real minimally coupled scalar has $X = Y = 1$. For N_S scalars, N_D Dirac fermions, N_V massless vectors and N_P Proca fields (taken in the limit $m \rightarrow 0$, i.e. a massless vector plus one scalar),

$$Y = N_S - 4N_D + 2N_V + 3N_P, \quad X_B = N_S + 2N_D - 4N_V - 3N_P. \quad (3)$$

The value of X depends on the matter-cutoff convention. Eq. (3) is the counting for a matter cutoff that is a function of the whole fluctuation operator $-\bar{D}^2 + E$ (“type B”, used in Refs. [8, 10]); for a cutoff that is a function of $-\bar{D}^2$ alone (“type A”) X changes and Y does not (Section 4.4).

Assumption 2.1 (Content and truncation). The QDG-like content is n_f Dirac fermions and the two QDG vectors, counted either as two Proca fields, $(X_B, Y) = (2n_f - 6, 6 - 4n_f)$, or as two massless vectors, $(2n_f - 8, 4 - 4n_f)$. The Weyl-gauged operators of QDG (dilaton, Weyl vector and axial vector with non-minimal couplings, distortion sector, finite masses) are not part of the flow, and gauge and parametrisation dependence were not studied: only $\alpha = 1$ and the linear split are used.

2.2 The model flow of the first version [MODEL]

The one-loop system of the first version is

$$\eta = -\frac{\tilde{G}A(\lambda)}{1 - \tilde{G}B(\lambda)} + C_\eta \tilde{G}, \quad \beta_{\tilde{G}} = (2 + \eta)\tilde{G}, \quad (4)$$

$$\beta_\lambda = -(2 - \eta)\lambda + \frac{\tilde{G}B(\lambda)}{2\pi} + C_{\text{bl}}\tilde{G} + \frac{72\lambda^2}{16\pi^2}, \quad (5)$$

$$A = \frac{5}{1 - 2\lambda} - \frac{3}{1 - \lambda} - 1 = \frac{1 + 4\lambda - 2\lambda^2}{D(\lambda)}, \quad B = A + 1 = \frac{2 + \lambda}{D(\lambda)}, \quad D = (1 - 2\lambda)(1 - \lambda),$$

with $C_\eta = (n_f - 4)/(3\pi)$ and $C_{\text{bl}} = (3 - 2n_f)/(2\pi)$. By the matching used in Eq. (1) these are $C_\eta = X/6\pi$ with $X = 2n_f - 8$ and $C_{\text{bl}} = Y/4\pi$ with $Y = 6 - 4n_f$, i.e. the curvature coefficient is that of two massless vectors and the volume coefficient that of two Proca fields [8]. We keep this mixed counting wherever we speak of “the model with its own coefficients”.

The system is a model, for the reasons given in the companion paper [8] and in Ref. [10]: A and B have the form of signed multiplicities of the volume term of the Wetterich trace, but are used as if they were the coefficient of R , which is what determines η ; the system has poles at $\lambda = \frac{1}{2}$ and $\lambda = 1$ whereas the derived flow has only $\lambda = \frac{1}{2}$; the term $72\lambda^2/16\pi^2$ is not derived from the action; and its pure-gravity limit is $(\tilde{G}_*, \lambda_*) = (0.3502, 0.0313)$, not the EH fixed point $(0.7073, 0.1932)$ (script `compare_flows.py`). No rescaling of $\tilde{G}, \lambda$ maps the model onto the derived flow. Whether it arises from another gauge or from the distortion sector is not known.

2.3 Admissible fixed points and the critical number [DERIVED]

Definition 2.2 (Admissible fixed point). A non-Gaussian zero $(\tilde{G}_*, \lambda_*)$ of $(\beta_{\tilde{G}}, \beta_\lambda)$ is *admissible* if $\tilde{G}_* > 0$, $\lambda_* < \frac{1}{2}$ and $\omega_* \equiv 1 - \tilde{G}_* B_*(\lambda_*) > 0$, where $B_* = B_2$ for the derived flow and $B_* = B$ for the model. The last condition keeps the fixed point on the same side of the pole of η as the Gaussian fixed point; it is a convention (it does not follow that ω stays positive along a trajectory), and it is the one used in the first version, in the companion paper and in Ref. [10].

Definition 2.3 (Critical number). For a one-parameter family of contents, n_{crit} is the supremum of the n_f for which an admissible fixed point exists. The statement “the fixed point exists only for $n_f < n_{\text{crit}}$ ” is a statement about this family, this flow and Definition 2.2; it is not a statement about the theory.

The first version restricted the search to $-2 < \lambda < 0.5$. We do not impose that window; where it matters we say so (Section 3).

3 The model flow: exact algebraic structure

3.1 The master polynomial [DERIVED]

With $\eta = -2$ (the condition $\beta_{\tilde{G}} = 0$ for $\tilde{G} \neq 0$), Eq. (4) multiplied by $1 - \tilde{G}B$ is the quadratic

$$C_\eta B \tilde{G}^2 + [A + 2B - C_\eta] \tilde{G} - 2 = 0, \quad (6)$$

and $\beta_\lambda = 0$ is linear in $\tilde{G}$:

$$\tilde{G} = \frac{N_L(\lambda)}{D_L(\lambda)}, \quad N_L = 4\lambda - \frac{9\lambda^2}{2\pi^2}, \quad D_L = \frac{B}{2\pi} + C_{\text{bl}}. \quad (7)$$

Write $A_n = 1 + 4\lambda - 2\lambda^2$, $B_n = 2 + \lambda$ (so $A = A_n/D$, $B = B_n/D$), $N_n = D N_L$ and $D_{Ln} = B_n/2\pi + C_{\text{bl}}D$ (so $\tilde{G} = N_n/D_{Ln}$). Substituting Eq. (7) into Eq. (6) and clearing denominators,

$$M(\lambda; n_f) \equiv C_\eta B_n N_n^2 + (A_n + 2B_n - C_\eta D) N_n D_{Ln} - 2D D_{Ln}^2 = 0. \quad (8)$$

Proposition 3.1 (Master polynomial). $M(\lambda; n_f) = D(\lambda) P(\lambda; n_f)/(12\pi^5)$ with zero remainder, where $P = \sum_{k=0}^7 c_k(n_f) \lambda^k$ has the coefficients of Appendix A. In particular

$$c_7 = 162(n_f - 4), \quad c_1 = 8\pi^3(2n_f - 5)(10n_f - 15\pi - 16), \quad c_0 = -6\pi^3(2n_f - 5)^2. \quad (9)$$

P has degree 7 for $n_f \neq 4$ and degree 6 for $n_f = 4$. These eight coefficients are exactly those printed in the first version.

Proof. Direct expansion of Eq. (8), carried out in exact rational arithmetic over $\mathbb{Q}[n_f, \pi, \pi^{-1}]$ (script `poly_exact.py`; no floating point). The polynomial M has degree 9; division by $D = 2\lambda^2 - 3\lambda + 1$ leaves no remainder, and the quotient multiplied by $12\pi^5$ coincides coefficient by coefficient with the first version's $c_0, \dots, c_7$ (transcribed from its Eqs. (16)–(23)). $\square$

Remark 3.2 (Degree and spurious factors). The first version attributed the drop from degree 9 to 7 to a cancellation of the λ^9 and λ^8 coefficients. The actual reason is the factor D : M has degree 9 and is divisible by $(1 - 2\lambda)(1 - \lambda)$, whose roots are the poles of A and B and are not fixed points. Every real root of P is a candidate only: one must check $D_L \neq 0$, $\tilde{G} = N_L/D_L > 0$, $\lambda < \frac{1}{2}$ and $\omega > 0$. At $n_f = \frac{5}{2}$ one has $D_L(0) = 0$ and $c_0 = c_1 = 0$: $\beta_\lambda(\tilde{G}, 0) = \tilde{G}(5 - 2n_f)/2\pi$ vanishes identically, the physical fixed point has $\lambda_* = 0$ exactly (with $\tilde{G}_* = 0.3974$; the second solution of Eq. (6) at $\lambda = 0$, $\tilde{G} = 15.81$, has $\omega < 0$), and the double root of P at $\lambda = 0$ contains both. This is a zero of λ_* , not a singularity of the branch.

3.2 Roots and the physical branch [NUMERICAL]

For $n_f = 3$ the polynomial P has seven *real* roots (script `model_flow_bound.py`):

λ	$\tilde{G} = N_L/D_L$	ω	status
−15.6646	362.49	+10.19	admissible, large coupling
−0.48112	5.149	−1.691	$\omega < 0$
−0.019360	0.4318	+0.1924	admissible, the fixed point of the first version
+0.11574	25.33	−77.86	$\omega < 0$
1.4399, 5.647, 8.364	—	—	$\lambda > \frac{1}{2}$ (beyond both poles)

The first version stated that only three roots are real and that “the other four are complex”; four are real, one of them at $\lambda = -15.66$ outside the window $(-2, 0.5)$ that was scanned, with $\tilde{G} > 0$ and $\omega > 0$. This large-coupling solution exists for $1.5 < n_f < 4$ (at $n_f \rightarrow 1.5$, where $C_{\text{bl}} = 0$ and $D_L(-2) = 0$, it tends to $\lambda = -2$ with $\tilde{G}_* \rightarrow \infty$; at $n_f \rightarrow 4$, where $c_7 \rightarrow 0$, it tends

to $\lambda \rightarrow -\infty$), has $\tilde{G}_* \geq 10^2$ throughout and, at $n_f = 3$, complex exponents $14.4 \pm 12.7i$. It is unrelated to the branch that matters for n_{crit} and we do not discuss it further.

The branch connected to the Gaussian-side fixed point at $n_f = 0$ is unique (one admissible root with $\lambda > -1$ for each of 2000 values of $n_f \in [0, n_{\text{crit}})$), with λ_* monotonically decreasing and $\tilde{G}_*$ monotonically increasing in n_f ; its values at $n_f = 0, \dots, 4$ are in Table 1. All numbers of the first version's Tables 1 and 2 are reproduced, and those of its Table 4, whose rows are $n_f = n_{\text{crit}} - \delta$ with $\delta = 10^{-2}, \dots, 10^{-5}$ (Table 1).

Table 1: The admissible fixed point of the model flow (mixed counting, $C_\eta = (n_f - 4)/3\pi$, $C_{\text{bl}} = (3 - 2n_f)/2\pi$) along the branch, and its approach to n_{crit} ($\delta = n_{\text{crit}} - n_f$). $\eta_{\text{grav}} \equiv -\tilde{G}A/\omega$ is the gravitational part of $\eta = -2$. Script `model_flow_bound.py`.

n_f	λ_*	$\tilde{G}_*$	ω_*	θ_1	θ_2	η_{grav}
0	0.0659	0.2964	0.2448	9.863	3.651	
1	0.0438	0.3278	0.2322	9.714	3.624	
2	0.0165	0.3699	0.2157	9.569	3.585	
3	-0.0194	0.4318	0.1924	9.446	3.519	
4	-0.0733	0.5411	0.1530	9.448	3.370	
$\delta = 1$	-0.0662	0.5254	0.1585	9.43	3.40	-1.994
$\delta = 10^{-1}$	-0.1686	0.7968	0.0662	11.04	2.70	-2.067
$\delta = 10^{-2}$	-0.2066	0.9288	0.0231	16.73	1.76	-2.087
$\delta = 10^{-3}$	-0.2192	0.9776	0.0072	34.2	0.895	-2.093
$\delta = 10^{-4}$	-0.2232	0.9938	0.0020	97.5	0.366	-2.095
$\delta = 10^{-5}$	-0.2244	0.9987	0.0004	423	0.148	-2.095

3.3 The critical number: closed form and mechanism [DERIVED]

Proposition 3.3 (Closed form of $n_{\text{crit}}^{\text{model}}$). *Let $\lambda_c = 1 - \sqrt{6}/2 = -0.22474\dots$ be the root of A with $\lambda < \frac{1}{2}$ (the other root, $1 + \sqrt{6}/2$, lies beyond the poles). Then $A(\lambda_c) = 0$ and $B(\lambda_c) = 1$. Suppose that along a continuous family of admissible fixed points of Eqs. (4)–(5) (parametrised by n_f , or by (C_η, C_{bl})) one has $\omega_* \rightarrow 0$ while $(\tilde{G}_*, \lambda_*)$ remain in a compact subset of $\{\tilde{G} > 0, \lambda < \frac{1}{2}\}$. Then $(\tilde{G}_*, \lambda_*) \rightarrow (1, \lambda_c)$ and*

$$C_{\text{bl}} = C_{\text{bl}}^* \equiv 4\lambda_c - \frac{1}{2\pi} - \frac{9\lambda_c^2}{2\pi^2} = -1.081164344371\dots,$$

$$\text{i.e. } Y = Y_c \equiv 4\pi C_{\text{bl}}^* = -13.586311846397\dots \quad (10)$$

The value of C_η plays no role. For the QDG-like family, $C_{\text{bl}} = (3 - 2n_f)/2\pi$, so

$$n_{\text{crit}}^{\text{model}} = \frac{3 - 2\pi C_{\text{bl}}^*}{2} = 2 - 4\pi\lambda_c + \frac{9}{2\pi}\lambda_c^2 = 4.896577961599297\dots \quad (11)$$

Proof. From $\beta_{\tilde{G}} = 0$, $\eta = -2$: $-\tilde{G}A = \omega(2 + C_\eta\tilde{G})$. If $\omega \rightarrow 0$ while $\tilde{G}$ stays in a compact subset of $(0, \infty)$, then $\tilde{G}A \rightarrow 0$ and hence $A \rightarrow 0$; so $\lambda \rightarrow \lambda_c$, $B \rightarrow 1$ and $\omega = 1 - \tilde{G}B \rightarrow 1 - \tilde{G}$, which vanishes only for $\tilde{G} \rightarrow 1$. With $\eta = -2$, $\beta_\lambda = 0$ reads $0 = -4\lambda + \tilde{G}B/2\pi + C_{\text{bl}}\tilde{G} + 9\lambda^2/2\pi^2$; at $(1, \lambda_c)$ this gives Eq. (10). The numerical values follow from $\lambda_c^2 = \frac{5}{2} - \sqrt{6}$. $\square$

The point $(\tilde{G}, \lambda) = (1, \lambda_c)$ is a point where η is of the form $0/0$: numerator A and denominator ω vanish together, η is discontinuous there, and the fixed-point equation $\eta = -2$ holds along it for every C_η (it fixes only the direction of approach, $-\tilde{G}A/\omega \rightarrow -2 - C_\eta$). Numerically (Table 1):

1. For $n_f < n_{\text{crit}}$ one admissible fixed point with $\lambda > -1$ exists; for $n_{\text{crit}} \leq n_f \leq 2000$ none exists in the whole region $\lambda < \frac{1}{2}$ (scan with step 10^{-3} up to $n_{\text{crit}} + 0.02$, 10^{-2} up to 60, 1 up to 2000), so the bound does not depend on the window $(-2, 0.5)$. Bisection on the existence of an admissible root gives 4.89657796159936, which differs from Eq. (11) by 6×10^{-14} . At $n_f = n_{\text{crit}}$ itself the fixed point sits at the singular point, where the flow is not defined; existence is therefore strict, $n_f < n_{\text{crit}}$. n_{crit} is also a root of the quadratic polynomial $n_f \mapsto P(\lambda_c; n_f)$ (the other root is 2.2664, where $\tilde{G} = -2/C_\eta$).
2. The root does not stop at $\omega = 0$ and does not move to $\tilde{G} < 0$. It crosses $\omega = 0$ linearly at $n_f = n_{\text{crit}}$ and continues, with $\omega < 0$ and $\tilde{G} \simeq 1.003$, until it collides with a second real root at $n_{\text{dbl}} = 4.8965822008$, i.e. 4.2×10^{-6} above n_{crit} , after which both are complex. The only real root of P left in $(-2, 0.5)$ is the unrelated one near $\lambda \simeq 0.27$ with $\tilde{G} < 0$ (the first version's Table 3, reproduced: $\lambda = 0.2699, 0.3042, 0.3739, 0.4703$ and $\tilde{G} = -26.8, -13.5, -4.59, -0.61$ at $n_f = 5, 6, 10, 50$).
3. The anomalous dimension does not diverge: $\eta = -2$ at every point of the branch, and its gravitational part tends to $-2 - C_\eta \tilde{G} = -2.095$. What diverges is the larger critical exponent, $\theta_1 \propto \omega_*^{-1}$ (from 9.4 at $\delta = 1$ to 423 at $\delta = 10^{-5}$), while the smaller one tends to zero ($\theta_2 = 0.148$ at $\delta = 10^{-5}$), as expected when a fixed point is about to collide with a partner.

The first version's description “ η_N^{grav} diverges at the critical point” and “beyond it the continued root moves to $\tilde{G} < 0$ ” are therefore not correct; the mechanism is the one stated in Proposition 3.3.

3.4 What the model bound constrains [DERIVED] / [NUMERICAL]

Proposition 3.3 shows that the bound of the model is a bound on C_{bl} , i.e. on the volume-term coefficient Y , and that C_η (the curvature-term coefficient X) does not enter the location of the singular point. Extending the model by $C_\eta = X/6\pi$ and $C_{\text{bl}} = Y/4\pi$ for arbitrary content (an extension that the model itself does not contain), the set of (X, Y) with an admissible fixed point with $\lambda > -2$ is bounded below by the horizontal line $Y = Y_c$ for every $X \geq X_* = 1.551$ (checked on $X \leq 120$; to 10^{-12} in Y); for $-10 \leq X < X_*$ the boundary lies at most 0.07 below Y_c , because the physical root then annihilates with its partner shortly before or after reaching the singular point (script `region_xy.py`). Hence the model's bound on the number of fermions depends on the counting essentially through $Y(n_f)$ alone (consistent contents, $C_\eta = X/6\pi$, $C_{\text{bl}} = Y/4\pi$):

content (n_f Dirac plus)	$Y(n_f)$	model bound $n_{\text{crit}}^{\text{model}}$
two Proca fields	$6 - 4n_f$	4.8966 (closed form)
two massless vectors	$4 - 4n_f$	4.3966 (end of branch: 4.39663)
nothing else	$-4n_f$	3.3966 (exact)

(Eq. (11) with C_{bl} replaced accordingly; the end of the branch coincides with it when the physical root is the one that reaches the singular point, which holds for the first and third rows, $X \geq X_*$ there; in the second row $X = 0.79 < X_*$ and the end of the branch lies 5×10^{-5} higher; script `model_flow_bound.py`).

Remark 3.4 (Standard-Model content). The first version called $n_f = 3$ “the Standard Model” and concluded that it is “safely below the bound”. Three Dirac fermions are not the Standard-Model fermions: the latter are 15 Weyl fermions per generation, 22.5 Dirac-equivalents for three generations, together with 12 gauge bosons and a Higgs doublet. With the minimal-coupling counting $(X, Y) = (1, -62)$ the extended model has no admissible fixed point at all, because $Y = -62 < Y_c$, whereas the derived flow has one (Table 6). The statement of the first version holds for $n_f = 3$ Dirac fermions with the two Proca fields, and for nothing else. Its quantification

as “1.9 fermion generations” is also not meaningful: n_f counts Dirac fields, and $n_{\text{crit}} - 3 = 1.8966$ is a number of Dirac fields.

4 The derived flow: the bound from X

4.1 Cubic and end of the branch [INPUT]

For the flow (1) the admissible fixed points are the points

$$\lambda_* = \frac{1-u}{2}, \quad \tilde{G}_* = \frac{2\pi}{N(u)}, \quad N(u) = \frac{4}{u^2} - \frac{5}{2u} + \frac{c}{6}, \quad c = 14 - X, \quad (12)$$

where $u > 0$ is a root of the cubic

$$2c u^3 - (2c + 54 - 3Y) u^2 + 118u - 48 = 0, \quad (13)$$

with $N(u) > 0$ and $N(u) - 2\pi B_2 = 3/u^2 - 5/(3u) + c/6 > 0$ (the last is $\omega_* > 0$; $\omega_* = (N - 2\pi B_2)/N$) [8, 10]. For n_f Dirac fermions and two Proca fields it reads $(40 - 4n_f)u^3 - (76 + 8n_f)u^2 + 118u - 48 = 0$. The values along the branch are in Table 2; for pure gravity $(\tilde{G}_*, \lambda_*) = (0.7073, 0.1932)$, $\theta = 1.475 \pm 3.043 i$.

Table 2: Admissible fixed point of the derived flow, n_f Dirac fermions and two Proca fields, type-B counting, optimised cutoff ($X = 2n_f - 6$, $Y = 6 - 4n_f$). Script `derived_flow_bound.py`; $\max |\beta_i| < 10^{-12}$ in every row.

n_f	$\tilde{G}_*$	λ_*	$\theta_{1,2}$
0	0.5093	0.2290	$2.171 \pm 3.263 i$
2	0.8362	0.1568	$1.226 \pm 2.531 i$
3	2.7319	-0.3179	2.207, 0.723
6	5.9848	-2.8594	2.146, 3.695
9	23.50	-17.60	2.276, 3.957
9.9	233.5	-193.1	2.293, 3.996
9.99	2334	-1948	2.294, 4.000

As $c \rightarrow 0^+$ and for $Y < 8$ one root escapes, $u \simeq (54 - 3Y)/2c$, with $\lambda_* \rightarrow -\infty$ and $\tilde{G}_* \simeq 2\pi/\{c[\frac{1}{6} - \frac{5}{54-3Y}]\} \rightarrow +\infty$ [8]; for the QDG-like family near $n_f = 10$ this gives $\lambda_*(10 - n_f) \rightarrow -19.5$ and $\tilde{G}_*(10 - n_f) \rightarrow 52\pi/7 = 23.34$ (checked at $10 - n_f = 10^{-1}, \dots, 10^{-4}$: $-19.31, \dots, -19.50$ and $23.354, \dots, 23.338$). The run-away is the end of the branch; whether it is the end of the fixed point depends on whether a second admissible root remains at $c \leq 0$. That is settled exactly at $X = 14$:

Proposition 4.1 (Existence at $X = 14$). *For $X = 14$ ($c = 0$) an admissible fixed point exists if and only if $Y \geq Y_0 \equiv 18 - 118^2/576 = -6.1736\dots$. Its coordinate is $u_* = 96/(118 + \sqrt{118^2 - 192(54 - 3Y)}) \leq 96/118 = 0.8136$.*

Proof. For $c = 0$ the cubic (13) becomes $(3Y - 54)u^2 + 118u - 48 = 0$. It has no real root iff its discriminant $118^2 - 192(54 - 3Y)$ is negative, i.e. $Y < Y_0$. Otherwise its smallest positive root is $u_* = 96/(118 + \sqrt{118^2 - 192(54 - 3Y)})$ (for $Y \geq 18$ it is the unique positive root), and $u_* \leq 96/118$. For such u_* , $N = (8 - 5u_*)/(2u_*^2) > 0$ because $u_* < 8/5$, and $N - 2\pi B_2 = (9 - 5u_*)/(3u_*^2) > 0$ because $u_* < 9/5$. $\square$

4.2 The bound for n_f Dirac fermions: 10 and 11 [NUMERICAL]

For the two QDG-like contents, $c = 0$ occurs at $n_f = 10$ (two Proca, $Y = -34$) and $n_f = 11$ (two massless vectors, $Y = -40$), and $Y < Y_0$ there, so by Proposition 4.1 nothing is left at $c = 0$. A scan of n_f from 10 to 200 in steps of 0.01 and from 200 to 5000 in steps of 1 finds no admissible root (from 11 for the massless counting); the last admissible point is at $n_f = 9.999$ (10.999). Thus

$$n_{\text{crit}}^{\text{derived}} = 10 \text{ (two Proca),} \quad 11 \text{ (two massless vectors),} \quad (14)$$

for the optimised cutoff with $a = 1$ and the type-B matter counting. More generally, for n_f Dirac fermions together with fixed numbers (N_S, N_V, N_P) of other fields the content moves on a line of slope $(2, -4)$ in the (X, Y) plane and $X = 14$ is reached at

$$n_f^B = 7 + 2N_V + \frac{3}{2}N_P - \frac{1}{2}N_S, \quad (15)$$

which is the end of the branch whenever $Y(n_f^B) < -553/36 = -15.36$ (Proposition 4.2 below); otherwise a second root survives. Table 3 shows both situations (script `derived_flow_bound.py`).

Table 3: Bound on n_f in the derived flow, type B, optimised cutoff, for n_f Dirac fermions plus the listed fields. “formula”: Eq. (15); “Y there”: Y at that value; “last”: last n_f (step 0.01) with an admissible fixed point.

other fields	formula	Y there	last
none	7	-28	6.99
2 Proca	10	-34	9.99
2 massless vectors	11	-40	10.99
12 vectors (SM gauge bosons)	31	-100	30.99
12 vectors + 4 scalars	29	-88	28.99
12 vectors + 4 scalars + 2 Proca	32	-94	31.99
10 scalars + 2 Proca	5	-4	5.58
20 scalars + 2 Proca	0	26	8.76

Two consequences. First, a number of fermions is a bound only at fixed other content: each massless vector adds 2 to the bound and each scalar subtracts $\frac{1}{2}$; the Standard-Model gauge and Higgs sector with N_D Dirac-equivalent fermions gives $N_D < 29$, i.e. $g < 58/15 = 3.87$ generations of 7.5 Dirac fermions (three generations: $(\bar{G}_*, \lambda_*) = (3.217, -4.393)$; four: no fixed point). Second, the last two rows show that “ $X < 14$ ” is not a bound on matter content as such: when Y is not very negative a second root survives beyond $X = 14$.

4.3 Existence region in the (X, Y) plane [DERIVED]/[NUMERICAL]

The fixed point depends on the content only through (X, Y) . For $X < 14$ an admissible fixed point exists at every one of 4010 tested points ($X \in \{13.9, 13, 10, 5, 0, -10, -50, -100, -500, -2000\}$, $Y \in [-2000, 2000]$); for $X \geq 14$ the existence region is bounded below by a curve $Y_{\min}(X)$ and on the right by $X_{\max}(Y)$ (Figure 1). The boundary has a closed form wherever the fixed point leaves through the pole of η :

Proposition 4.2 (Exit through the pole of η). *Let $s = 1/u$. Suppose a family of admissible fixed points of the derived flow approaches $\omega_* = 0$ at finite coupling ($N > 0$). Then its content lies on the arc*

$$X(s) = 14 - 10s + 18s^2, \quad Y(s) = \frac{34}{3} - \frac{62}{3}s + 4s^2, \quad s > \frac{5}{6}, \quad (16)$$

and conversely each point of the arc has the fixed point $u = 1/s$, $\lambda_* = (1 - u)/2$, $\tilde{G}_* = 2\pi u^2/(1 - 5u/6)$ with $\omega_* = 0$. The minimum of Y along the arc is $Y = -553/36 = -15.3611\dots$, at $s = 31/12$, $X = 2599/24 = 108.29\dots$, with $(\tilde{G}_*, \lambda_*) = (1.390, 0.3065)$.

Proof. At a fixed point $\omega = (N - 2\pi B_2)/N$ with $N > 0$, so $\omega \rightarrow 0$ at finite coupling iff $N - 2\pi B_2 = 3/u^2 - 5/(3u) + c/6 = 0$, i.e. $c = 10s - 18s^2$, i.e. $X = 14 - 10s + 18s^2$. Then $2cu^2 = 20u - 36$ and $2cu^3 = 20u^2 - 36u$, and Eq. (13) becomes $3Yu^2 = 34u^2 - 62u + 12$, which is the second of Eqs. (16). On the arc $N = 2\pi B_2 = 1/u^2 - 5/(6u) > 0$ iff $u < \frac{6}{5}$, i.e. $s > \frac{5}{6}$, and $\tilde{G}_* = 2\pi/N$. Finally $dY/ds = -\frac{62}{3} + 8s = 0$ at $s = \frac{31}{12}$. $\square$

The numerical existence boundary (bisection on the exact cubic) coincides with this arc to the digits quoted for every $X \geq 38$ tested (Table 4); for $14 < X < X_j = 37.84$ it is instead a saddle-node arc (a double root of the cubic), which starts at $(14, Y_0)$ and meets the pole arc at $s = (77 + \sqrt{4009})/96$, the solution of $48s^2 - 77s + 10 = 0$ where the cubic has a double root exactly at $\omega = 0$ ($X_j = 37.84$, $Y_j = -10.33$). The right edge is the other branch of the same arc: $X_{\max}(Y) = X(s_+)$ with $s_+ = [\frac{62}{3} + \sqrt{(\frac{62}{3})^2 - 16(\frac{34}{3} - Y)}]/8$, which gives 277.4, 340.1, 431.4, 498.1, 591.9 for $Y = -6, 0, 10, 18, 30$ (numerical scan, step 0.1: 277.3, 340.0, 431.4, 498.1, 591.9). Combined with Proposition 4.1:

1. for $Y < -553/36$ there is no admissible fixed point for any $X \geq 14$ (grid scan to $X = 1000$ at $Y = -15.4, -16, -30, -100$; at $Y = -15.36$ one still exists), so the boundary of the existence region is exactly the line $X = 14$ and Eq. (15) holds;
2. for $-553/36 < Y < Y_0$ the region for $X > 14$ is a separate part of the plane, detached from the one at $X < 14$;
3. the largest X with a fixed point is bounded (for example $X_{\max} \simeq 340$ at $Y = 0$), so “ $X < 14$ ” is a statement about contents with sufficiently negative Y , i.e. with many fermions.

Existence is meant here without the requirement $\text{Re } \theta_i > 0$; near the boundary for $X > 14$ the exponents can have negative real part [10].

Table 4: Lower boundary $Y_{\min}(X)$ of the existence region for $X > 14$ (exact cubic, bisection) and the pole arc (16). Script `boundary_curve.py`. Below $X_j = 37.84$ the boundary is the saddle-node arc.

X	$Y_{\min}$ (numerical)	$Y(s)$, Eq. (16)
14^+	-6.1736	—
30	-8.754	(-8.363)
38	-10.362	-10.362
50	-12.373	-12.373
80	-14.812	-14.812
108.29	-15.361	-15.361
200	-11.968	-11.968
300	-3.934	-3.934
400	6.421	6.421

4.4 The matter-cutoff convention [DERIVED] / [NUMERICAL]

For a field with fluctuation operator $-\bar{D}^2 + E$ on a bundle of fibre dimension N , $\text{tr } b_2 = \frac{R}{6}N - \text{tr } E$. If the cutoff is a function of the whole operator (type B), E is absorbed in the argument of the threshold functions and the R -term of a field of Wetterich weight s is proportional to

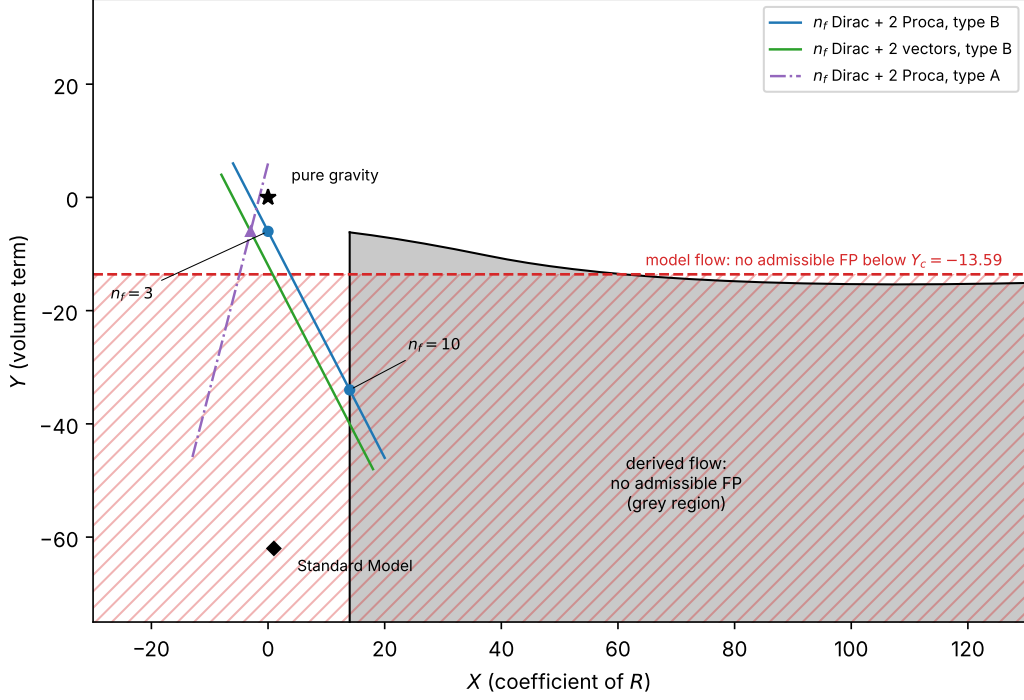


Figure 1: The (X, Y) plane. Grey: no admissible fixed point of the derived flow (optimised cutoff, type B); its boundary is the line $X = 14$ for $Y < -15.36$, then the saddle-node arc, then the pole arc (16). Hatched below the dashed line: no admissible fixed point of the model flow extended by $C_\eta = X/6\pi$, $C_{bl} = Y/4\pi$ ($Y < Y_c = -13.586$; for $X \gtrsim 1.55$ the boundary is exactly this line). Lines: n_f Dirac fermions with two Proca fields (type B, blue; dots: $n_f = 3, 10$), with two massless vectors (type B, green), and with two Proca fields in type A (purple; the triangle is $n_f = 3$). The Standard Model with minimal coupling ($X = 1, Y = -62$) lies in the model's excluded region and in the derived flow's allowed one. Script `region_xy.py`.

$s(N/6 - \text{tr}E/R) \Phi_1^1(0)$. If it is a function of $-\bar{D}^2$ alone (type A), E stays in the propagator and the same term is proportional to $s[N/6 \Phi_1^1(0) - (\text{tr}E/R) \Phi_2^2(0)]$. Normalising to a real scalar and writing $r = \Phi_2^2(0)/\Phi_1^1(0)$ ($r = \frac{1}{2}$ for the optimised cutoff with $a = 1$),

$$X_D^A = 6r - 4, \quad X_V^A = 2 - 6r, \quad X_P^A = 3 - 6r, \quad X_S = 1, \quad Y \text{ unchanged}; \quad (17)$$

type B is the formal value $r = 1$ ($X_D = 2$, $X_V = -4$, $X_P = -3$). For $r = \frac{1}{2}$, $X_D^A = X_V^A = -1$ and $X_P^A = 0$, so that for n_f Dirac fermions and two Proca fields $X^A = -n_f$ and $c = 14 + n_f > 0$: the condition $c \rightarrow 0$ is never reached. Eq. (17) for a general profile is derived here from the heat-kernel expansion; the formula for the optimised cutoff reproduces the companion paper's type-A values. The scan (script `derived_flow_bound.py`) finds an admissible fixed point at every one of 29 801 values of $n_f \in [0, 10^4]$, for both the Proca and the massless-vector counting. Representative points (Proca, $a = 1$):

n_f	$\tilde{G}_*$	λ_*	$\theta_{1,2}$
3	1.819	-0.113	1.491, 0.978
10	1.738	-1.326	1.955, 3.722
100	0.336	-2.672	2.006, 3.982
1000	0.0372	-2.964	2.001, 3.998

At large n_f the cubic tends to $u^3 - 7u^2 = 0$, so $\lambda_* \rightarrow -3$, $\tilde{G}_* \simeq 12\pi/n_f \rightarrow 0$ and $\theta \rightarrow (2, 4)$. In type A the number of fermions is not bounded by the existence of the fixed point, in the

cases tested. We do not know which convention corresponds to the matter literature [6] (whose abstract states that, within certain approximations, a given number of gauge fields allows only a maximal number of scalar and fermion degrees of freedom compatible with a fixed point at positive Newton coupling, a qualitative statement in line with the type-B result and not with the type-A one for this content). The bound (14) is, in this sense, a property of the type-B convention.

4.5 Regulator robustness [NUMERICAL]

We repeated the scan for the optimised profile $R^{(0)} = a(1-z)\theta(1-z)$, $a = 0.5, 0.75, 1, 1.5, 2, 3$, and for the exponential profile $R^{(0)} = a z/(e^z - 1)$, $a = 0.5, 1, 2$, with the threshold functions computed by Gauss–Legendre quadrature (scripts `rre.py`, `rre_run50.py` of Ref. [10], unchanged; `regulators.py`), widening the λ -window of the scan from $[-80, \cdot)$ to $[-3000, \cdot)$ because the fixed point runs off to large $|\lambda_*|$ near the end. Table 5 gives the last n_f on a grid of step 0.1 with an admissible fixed point (in all cases the exponents also have $\text{Re } \theta_i > 0$), and the type-A fixed point at $n_f = 3$ with the profile-dependent X^A of Eq. (17).

Table 5: Nine regulators. $r = \Phi_2^2(0)/\Phi_1^1(0)$. “last”: last n_f (step 0.1) with an admissible fixed point, type-B counting. “A, $n_f = 3$ ”: fixed point $(\tilde{G}_*, \lambda_*)$ in type A (Proca). Script `regulators.py`.

profile	a	r	last, 2 Proca	last, 2 vectors	A, $n_f = 3$
optimised	0.5	0.557	10.3	11.3	(2.03, −0.04)
optimised	0.75	0.524	10.1	11.1	(1.90, −0.08)
optimised	1	0.500	9.9	10.9	(1.82, −0.11)
optimised	1.5	0.466	9.7	10.7	(1.70, −0.17)
optimised	2	0.443	9.6	10.6	(1.62, −0.21)
optimised	3	0.410	9.4	10.4	(1.51, −0.27)
exponential	0.5	0.652	10.8	11.8	(0.79, 0.07)
exponential	1	0.608	10.6	11.6	(0.62, 0.04)
exponential	2	0.562	10.3	11.3	(0.51, −0.04)

(i) In type B, an admissible fixed point exists for all nine regulators up to $n_f = 9.4$; the fraction of regulators with one falls through 0.89 (9.5) to 0 (10.9) for two Proca fields, and from 1 (10.1) to 0 (11.3) for two massless vectors. The end of the branch therefore lies between 9.4 and 10.8 (two Proca) and between 10.4 and 11.8 (two vectors), and for the optimised cutoff with $a = 1$ the grid value 9.9 is consistent with the exact 10. At the last point the fixed point is far out: $(\tilde{G}_*, \lambda_*) = (233, -193)$ for the optimised profile with $a = 1$, $(737, -389)$ for $a = 0.5$ and $(127, -917)$ for the exponential profile with $a = 2$ (two Proca). (ii) In type A (with X^A from Eq. (17)) all nine regulators have an admissible fixed point at each of $n_f = 1, 3, 6, 10, 12, 15, 20, 30, 50$ tested, with exponents of positive real part. (iii) The existence of an end of the branch is thus regulator-robust within type B and absent in type A; its position is regulator-dependent at the $\pm 10\%$ level. Existence on a finite range of n_f is a necessary condition only [10]; it says nothing about the position or the exponents. Regulator dependence is not defined for the model flow: its functions A , B are fixed rational functions, not threshold functions of a profile.

4.6 Reliability and range [NUMERICAL]

For the derived flow with two Proca fields the fixed point at $n_f = 3$ is $(\tilde{G}_*, \lambda_*) = (2.73, -0.318)$ with real exponents in type B and $(1.82, -0.11)$ in type A (Table 5); in type B the nine regulators put $\tilde{G}_*$ between 0.58 and 2.83, the exponential ones with a complex pair of exponents [8]. By

the convention $\tilde{G}_* < 2$, $\lambda_* > -1$ for where an expansion in R/k^2 is plausible (a convention, not a derived bound), the type-B optimised-cutoff fixed point is already outside the range at $n_f = 3$, $|\lambda_*| > 5$ for $n_f \gtrsim 7$, and at $n_f = 9.9$ it is at $(233, -193)$. The bound (14) is therefore not the edge of a range in which the truncation is informative: it is where a fixed point already far outside that range ceases to exist.

5 Comparison of the two bounds

Table 6 gives the same contents in the two flows (the model extended by $C_\eta = X/6\pi$, $C_{\text{bl}} = Y/4\pi$), and Table 7 the resulting bounds.

Table 6: Same content, two flows (optimised cutoff, type B; model: admissible fixed point with $\lambda > -2$; without that window the model has in addition a large-coupling solution for $(-5, -56)$, at $\lambda = -51.8$, $\tilde{G} = 321$). Script `compare_flows.py`.

content	(X, Y)	derived flow $(\tilde{G}_*, \lambda_*)$, θ	model flow $(\tilde{G}_*, \lambda_*)$, θ
pure gravity	(0, 0)	(0.707, 0.193), $1.475 \pm 3.04 i$	(0.350, 0.031), 10.39, 3.60
$n_f = 3 + 2$ Proca	(0, -6)	(2.732, -0.318), 2.21, 0.72	(0.434, -0.020), 9.80, 3.52
$n_f = 4 + 2$ Proca	(2, -10)	(3.904, -1.068), 1.80, 3.16	(0.545, -0.074), 9.90, 3.37
$n_f = 5 + 2$ Proca	(4, -14)	(4.783, -1.820), 2.04, 3.50	none
$n_f = 6 + 2$ Proca	(6, -18)	(5.985, -2.859), 2.15, 3.70	none
Standard Model, minimal coupling	(1, -62)	(3.217, -4.393), 2.08, 3.93	none
Standard Model + 2 Proca	(-5, -56)	(2.187, -2.693), 2.04, 3.88	none
SM with four generations	(16, -92)	none	none

Table 7: Bounds on n_f (Dirac fermions) from the existence of an admissible fixed point; all conditional on the flow, the counting of the two QDG vectors and the convention.

flow	what is constrained	2 Proca	2 massless vectors
model (first version)	volume $Y \geq -13.586$	4.8966	4.3966
derived, type B, Litim $a = 1$	coefficient, curvature coefficient, $X < 14$	10	11
derived, type B, nine regulators	same, regulator-dependent	9.4–10.8	10.4–11.8
derived, type A	—	no end up to 10^4	no end up to 10^4

The two bounds are not two approximations of one number. They constrain different coefficients (Y from below in the model, X from above in the derived flow), they arise through different mechanisms (an isolated singular point of η in the model; a run-away at $c = 0$, a saddle-node and a curve of singular points of η in the derived flow, Propositions 3.3–4.2), and they give different answers for part of the contents of Table 6: the Standard-Model content, and $n_f = 5, 6$ with two Proca fields, have a fixed point in the derived flow and none in the extended model. The flows share one exit: an admissible fixed point can leave the admissible set at finite coupling by reaching a point where η is 0/0 (the isolated point $(1, \lambda_c)$ in the model, the arc (16) in the derived flow). The derived flow has two further exits, the run-away at $c = 0$ and the saddle-node arc. The first exit is a feature of how Definition 2.2 treats the pole, not a property of any theory.

Remark 5.1 (Other truncations; transcribed, not recomputed). In the R^2 truncation of Lauscher and Reuter [5], extended to Standard-Model-like matter in the chapter on regulator-robust existence of Ref. [10], the fixed point followed in N_D at fixed exponential-family parameter $s = 2$

($N_S = 4$, $N_V = 12$) is followed up to $N_D = 28$ and lost at $N_D = 28.5$ with λ_* and $\tilde{G}_*$ growing rapidly; that chapter does not claim an absence proof. These numbers are transcribed from there (the scripts need a symbolic package that is not available here) and are not recomputed. For the same sector the EH flow of this paper gives $N_D < 29$ for the optimised profile with $a = 1$ (exact; the last grid point with a fixed point is 28.5 in steps of 0.5) and 28–29.5 for the nine regulators (script `regulators_sm.py`). The profiles differ, so this is only an indication that a bound of the same size appears in both truncations for Standard-Model-like matter; nothing is computed here for the QDG-like content in the R^2 truncation.

5.1 Status of the claims of the first version

Table 8: Claims of the first version and their status.

Claim of the first version	Status	Comment
$P(\lambda; n_f)$ has degree 7 with the listed coefficients	Confirmed	exact, Proposition 3.1; reason for the degree: factor D
$n_{\text{crit}} = 4.896577961599$	Confirmed for the model	closed form, Eq. (11)
Physical root exists iff $n_f \leq n_{\text{crit}}$	Modified	strict inequality ($\omega = 0$ at n_{crit}); also true without the window $(-2, 0.5)$
“the other four roots are complex”	Wrong	all seven roots are real
η_N^{grav} diverges at n_{crit}	Wrong	$\eta = -2$ throughout; $\theta_1 \rightarrow \infty$, $\theta_2 \rightarrow 0$
Beyond n_{crit} the root moves to $\tilde{G} < 0$	Wrong	it continues with $\omega < 0$ and collides at n_{dbl}
“QDG is asymptotically safe $\iff n_f \leq 4.90$ ”	Withdrawn	property of a model; derived flow gives 10 (type B) or no bound (type A)
Standard Model ($n_f = 3$) is “safely below the bound”	Withdrawn	$n_f = 3$ is not the SM content (Remark 3.4)
Margin “1.9 fermion generations”	Withdrawn	1.9 Dirac fields
In EH gravity with matter the bound is absent, QDG’s is stronger	Withdrawn	the derived flow has a bound of its own ($X < 14$)
Bound specific to Litim regulator, others not studied	Replaced	nine regulators, Section 4.5

5.2 Open problems

Open Problem 5.2 (Weyl-gauged sector). Compute the contribution of the dilaton, the Weyl vector and the axial vector with their non-minimal couplings and masses, and of the distortion sector, to η and β_λ instead of the minimally coupled Proca counting. Only then can a bound on the QDG matter content be stated. Such terms shift (X, Y) ; Figure 1 and Table 4 show how large a shift is needed to cross each boundary.

Open Problem 5.3 (Origin of the model). Decide whether the functions A, B and the term $72\lambda^2/16\pi^2$ of the model can be obtained from the Hessians of some gauge or of the distortion sector. If they cannot, the bound 4.8966 has no bearing on QDG.

Open Problem 5.4 (Matter-cutoff convention). Fix the matter-cutoff convention by a principle, and decide whether the existence of an end of the branch (type B) or its absence (type A) is the property of gravity coupled to matter in the matter literature.

Open Problem 5.5 (Truncation, gauge and parametrisation). Repeat the analysis with higher-curvature operators for QDG-like content, for $\alpha \neq 1$ (including $\alpha \rightarrow 0$) and for an exponential parametrisation of the metric. All statements above are for the Einstein–Hilbert truncation, $\alpha = 1$ and the linear split.

6 Conclusions

The algebraic result of the first version stands: eliminating $\tilde{G}$ from the fixed-point equations of its one-loop system gives a degree-7 polynomial whose physical root ceases to be admissible at $n_f = 4.896577961599$, which is the closed form $2 - 4\pi\lambda_c + 9\lambda_c^2/2\pi$ with $\lambda_c = 1 - \sqrt{6}/2$ and is reached through an isolated singular point of the flow. What does not stand is its interpretation. The one-loop system is a model whose bound constrains the volume-term coefficient of the matter, $Y \geq -13.586$. In the derived Einstein–Hilbert flow the bound is on the curvature-term coefficient, $X < 14$, and gives 10 Dirac fermions with two Proca fields (11 with two massless vectors) for the optimised cutoff and the type-B matter counting; the existence region in the (X, Y) plane has a closed-form boundary; nine regulators move the end of the branch by about $\pm 10\%$; and with a matter cutoff that is a function of $-\bar{D}^2$ only no bound exists. These are statements about truncated flows with minimally coupled matter in a fixed gauge. The Weyl-gauged operators of QDG are in neither, so the present paper contains no bound on the matter content of QDG.

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A Coefficients of the master polynomial

The polynomial of Proposition 3.1, $P(\lambda; n_f) = \sum_{k=0}^7 c_k \lambda^k$ with $c_k = A_k n_f^2 + B_k n_f + C_k$, is (the list is generated from the exact computation by `gen_coeff_tex.py`; it coincides with Eqs. (16)–(23) of the first version):

	$A_k (n_f^2)$	$B_k (n_f)$	$C_k (n_f^0)$
c_7	0	162	−648
c_6	−72 π	−504 $\pi^2 + 396\pi + 81$	1476 $\pi^2 - 432\pi - 324$
c_5	64 $\pi^3 + 216\pi$	320 $\pi^4 - 352\pi^3 + 828\pi^2 - 1170\pi - 405$	−800 $\pi^4 + 384\pi^3 - 828\pi^2 + 1224\pi + 1620$
c_4	−288 $\pi^3 - 234\pi$	−800 $\pi^4 + 1328\pi^3 + 180\pi^2 + 1296\pi + 162$	992 $\pi^4 - 1304\pi^3 - 2124\pi^2 - 1440\pi - 648$
c_3	496 $\pi^3 + 108\pi$	160 $\pi^4 - 1968\pi^3 - 774\pi^2 - 639\pi$	608 $\pi^4 + 1856\pi^3 + 1422\pi^2 + 828\pi$
c_2	−408 $\pi^3 - 18\pi$	560 $\pi^4 + 1528\pi^3 + 270\pi^2 + 117\pi$	−752 $\pi^4 - 1480\pi^3 - 675\pi^2 - 180\pi$
c_1	160 π^3	−240 $\pi^4 - 656\pi^3$	600 $\pi^4 + 640\pi^3$
c_0	−24 π^3	120 π^3	−150 π^3

For $n_f = 3$ the polynomial has the seven real roots listed in Section 3.

B Scripts and core code

Every number in this paper is reproduced by one of the following scripts (NumPy and Matplotlib only; outputs stored next to them as *.out).

- `poly_exact.py`: exact elimination, $M = DP/12\pi^5$, comparison with the eight coefficients of the first version; `gen_coeff_tex.py`: Appendix A.
- `model_flow_bound.py`: roots, Table 1, closed form of $n_{\text{crit}}^{\text{model}}$, bisection with and without window, n_{dbl} , behaviour near n_{crit} , variants of C_{bl} , large-coupling roots.
- `derived_flow_bound.py`: derived flow, Tables 2, 3, scans beyond $c = 0$, type A, Standard-Model generations.
- `boundary_curve.py`: Propositions 4.1, 4.2, Table 4; `region_xy.py`: existence regions and Figure 1.
- `regulators.py`, `regulators_sm.py` (with `rre.py`, `rre_run50.py` of the book, unchanged): Table 5, Remark 5.1; `compare_flows.py`: Table 6.

The core of the derived-flow search and of the model polynomial is:

```
import numpy as np
from numpy.polynomial import polynomial as Pl
pi = np.pi

def derived_fps(X, Y):
    # Eqs. (fp-closed), (cubic); u = 1 - 2*
    lambda
    c = 14 - X; out = []
    coeffs = np.array([2*c, -(2*c + 54 - 3*Y), 118., -48.])
    if abs(c) < 1e-14: coeffs = coeffs[1:]
    for u in np.roots(coeffs):
        if abs(u.imag) < 1e-9*max(1, abs(u)) and u.real > 0:
            u = u.real; N = 4/u**2 - 2.5/u + c/6
            if N > 0 and 3/u**2 - 5/(3*u) + c/6 > 0:
                out.append((2*pi/N, (1 - u)/2)) # (G*, lambda*)
    return out

D = np.array([1, -3, 2.]) # (1-2l)(1-l)
def model_P(nf):
    # P/(12 pi^5) = M/D, Eq. (M)
    Ceta, Cbl = (nf - 4)/(3*pi), (3 - 2*nf)/(2*pi)
    Bn, An = np.array([2, 1.]), np.array([1, 4, -2.])
    Nn = Pl.polymul(np.array([0, 4, -9/(2*pi**2)]), D)
    DLn = Pl.polyadd(Bn/(2*pi), Cbl*D)
    t1 = Ceta*Pl.polymul(Bn, Pl.polymul(Nn, Nn))
    t2 = Pl.polymul(Pl.polysub(Pl.polyadd(An, 2*Bn), Ceta*D), Pl.polymul(Nn, DLn))
    t3 = 2*Pl.polymul(D, Pl.polymul(DLn, DLn))
    return Pl.polydiv(Pl.polysub(Pl.polyadd(t1, t2), t3), D)[0]

print(derived_fps(0, -6)) # n_f = 3, two Proca: [(2.7319,
    -0.3179)]
lc = 1 - np.sqrt(6)/2
print(2 - 4*pi*lc + 9*lc**2/(2*pi)) # 4.896577961599...
```

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