

The Spectral Action and the Wetterich Equation in Quantum Distortion Gravity: What the Dirac Operator Can and Cannot Generate

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Abstract

We compare the spectral action of the Dirac operator of Quantum Distortion Gravity (QDG) with the Wetterich functional renormalization group (FRG) equation. In QDG the fermion is coupled to the metric, to the dilaton (through the Yukawa term) and to the axial vector $\tilde{a}_\mu$ of the distortion, but *not* to the Weyl gauge vector V_μ : the Dirac kinetic term is already Weyl invariant, and a V_μ coupling would be anti-Hermitian. Four results follow. (i) We derive the Laplace-type form of the squared Dirac operator with axial and scalar couplings, and obtain the Seeley–DeWitt coefficients a_2 and a_4 (verified by exact operator identities, a quantum-electrodynamics benchmark and the Dirac conformal-anomaly pair $(c, a) = (6/120, 11/360)$); no coefficient contains V_μ , so the fermionic spectral action cannot generate a kinetic or mass term for the Weyl vector, and the kinetic term of $\tilde{a}_\mu$ is generated with the same coefficient as for a vector coupling. (ii) With the cutoff scale replaced by the dilaton, the Dirac operator is Weyl covariant up to a similarity transformation, so $\text{Tr} f(\mathcal{D}_\phi^2/\kappa_0^2)$ is exactly Weyl invariant; it equals the ordinary spectral action of the unitary-gauge metric $\hat{g} = (\phi/\phi_0)^2 g$ and depends on V_μ only through its pure-gauge value. (iii) For a Litim-type fermionic cutoff the fermion block of the Wetterich flow equals minus the spectral counting function $\text{Tr} \theta(k^2 - \mathcal{D}^2)$; with the heat-kernel coefficients this reproduces the Dirac contributions $(X, Y) = (2, -4)$ to the Einstein–Hilbert flow of the QDG monograph, and similarly for scalars and vectors. (iv) We state what is not established: a spectral origin of the Weyl vector sector, of the graviton block, and of the Proca masses. The fixed-point structure of the flow plays no role here.

Keywords: quantum gravity, metric-affine gravity, Weyl symmetry, functional renormalization group, spectral action, Seeley–DeWitt coefficients, Dirac operator

1 Introduction

Two constructions are often discussed side by side as routes to the ultraviolet structure of gravity: the spectral action of Connes and Chamseddine [4, 5], in which the bosonic action is the trace of a cutoff function of the square of a Dirac operator, and the functional renormalization group (FRG) based on the Wetterich equation [7]. Both are heat-kernel constructions: the coefficients of the Seeley–DeWitt expansion [8] of the relevant fluctuation operators enter the spectral action with the moments of the cutoff function, and enter the FRG flow with the threshold functions of the regulator.

This note asks what this common structure gives for Quantum Distortion Gravity (QDG) [1, 2], a metric-affine theory with gauged Weyl symmetry, in which the independent connection

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contributes the distortion tensor $D^\rho_{\mu\nu}$, the Weyl gauge vector V_μ and the dilaton ϕ compensate local rescalings, and the fermion couples to the geometry through the Levi-Civita spin connection, the dilaton and the axial vector $\tilde{a}_\mu$ of the distortion. We do not propose a unification. We work out, with scripts, what the QDG Dirac operator does and does not generate, and in which sense its spectral action and the fermion block of the Wetterich trace are the same object.

Summary of results.

1. [DERIVED] [SCRIPT] The Dirac operator of QDG does not involve V_μ (Section 2). Its square is of Laplace type with an effective connection and an endomorphism that we compute (Proposition 3.1); the coefficients a_2, a_4 are given in Proposition 3.2. V_μ appears in none of them.
2. [DERIVED] A Weyl-invariant spectral action exists, with the dilaton playing the role of the cutoff scale (Proposition 4.2); it coincides with the spectral action of the unitary-gauge metric, and in QDG variables it has exactly the structure of the Weyl-invariant ultraviolet action with V_μ replaced by $\partial_\mu \ln \phi$.
3. [DERIVED] [SCRIPT] For the fermion block the Wetterich flow with a Litim-type cutoff is the spectral counting function (Proposition 5.1); the monograph's matter coefficients (X, Y) for scalars, Dirac fermions and vectors are reproduced (Table 1).
4. [OPEN] Not established: a spectral origin of the kinetic term of the gauge-invariant massive vector $\hat{V}_\mu$ (unitary gauge), of the graviton block, and of the Proca masses; the spectral action of the physical Lorentzian axial coupling (Section 4).

Logical status. [STANDARD]: standard, source cited. [DERIVED]: derived here. [SCRIPT]: reproduced by a script of `code/` (numpy only). [ASSUMPTION]: stated assumption. [OPEN]: open. *Proposition* is proved under the stated hypotheses; *Remark* is not.

Conventions. Section 2 uses the QDG monograph's conventions [2]: signature $(-, +, +, +)$, $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$, a field of Weyl weight w scales as Ω^w under $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $V_\mu \rightarrow V_\mu - \partial_\mu \omega$ ($\omega = \ln \Omega$), $D_\mu \Phi = (\partial_\mu + wV_\mu)\Phi$. Everything from Section 3 on is Euclidean, with Hermitian γ^μ , $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$, $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, and the curvature sign such that the Ricci scalar of a sphere is positive.

2 The Dirac operator of QDG

2.1 What couples to the fermion

The ultraviolet action of QDG contains the fermion sector [2]

$$S_\psi = \int d^4x \sqrt{-g} \left[\bar{\psi} (i\gamma^\mu \hat{\nabla}_\mu - y\phi) \psi + \frac{3}{2} \tilde{a}_\sigma \bar{\psi} \gamma^\sigma \gamma_5 \psi \right], \quad (1)$$

with $\hat{\nabla}_\mu \psi = \partial_\mu \psi - \frac{1}{4} \hat{\omega}_{\mu ab} \gamma^{ab} \psi$ the Levi-Civita spin connection, $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon^{0123} = +1$. The weights are $w_\psi = -3/2$, $w_\phi = -1$, $w_{\tilde{a}} = 0$ (index down). Three statements, derived in the monograph and re-run for this note (script `partI_II_audit_fermion.py`, output `partI_II_audit_fermion.out`, 17 checks, 0 failures) [DERIVED] [SCRIPT], fix the content of (1):

1. Writing the general metric-affine spin connection as $\omega_{\mu ab} = \hat{\omega}_{\mu ab} + D_{a\mu b}$, only the part antisymmetric in ab acts on a spinor. Of the irreducible pieces of the distortion, the

traceless tensor $\hat{q}$ and the vector Q drop out, the vector $P - R$ couples through an anti-Hermitian term that is absent from the real action, and the axial vector $\tilde{a}_\sigma$ couples with strength exactly $\frac{3}{2}$: $-\frac{i}{4}\gamma^c\gamma^{ab}D_{acb} = \frac{3}{2}\gamma_\sigma\gamma_5\tilde{a}^\sigma$ for $D_{abc} = \varepsilon_{abco}\tilde{a}^\sigma$.

2. The Dirac kinetic term with the Levi-Civita spin connection is invariant under $\psi \rightarrow \Omega^{-3/2}\psi$ without any further field (because $\gamma_c\gamma^{cb} = -3\gamma^b$ and the shift of the spin connection cancels the derivative of $\Omega^{-3/2}$).
3. Hence V_μ does not couple to ψ . A term $-\frac{3}{2}iV_\mu\bar{\psi}\gamma^\mu\psi$ would be anti-Hermitian.

Proposition 2.1 (Coupling equals weight). *Let X be a field of weight w and $D_\mu X = (\partial_\mu + cV_\mu)X$ with constant c . Then $\delta(D_\mu X) = w\omega D_\mu X$ if and only if $c = w$.*

Proof. With $\delta X = w\omega X$, $\delta V_\mu = -\partial_\mu\omega$: $\delta(\partial_\mu X) = w(\partial_\mu\omega)X + w\omega\partial_\mu X$ and $\delta(cV_\mu X) = -c(\partial_\mu\omega)X + cw\omega V_\mu X$, so $\delta(D_\mu X) = w\omega D_\mu X + (w - c)(\partial_\mu\omega)X$. $\square$

Remark 2.2 (Why V_μ is absent from the Dirac operator). Proposition 2.1 says that if a derivative $\partial_\mu + cV_\mu$ of ψ were used, covariance would require $c = w_\psi = -3/2$. For a spinor, however, the derivative that appears in the Dirac operator is the spin-covariant one, whose Weyl shift already supplies the inhomogeneous term (statement 2 above); adding $-\frac{3}{2}V_\mu$ would be neither needed nor allowed by reality of the action (statement 3). This agrees with the monograph and with the fact that the one-loop running of the Weyl gauge coupling has no fermionic contribution. The sign and value $c = -3/2$ is, for the record, the one that a covariant derivative of a weight $-3/2$ field would have in the dilaton-like convention $D_\mu\phi = (\partial_\mu - V_\mu)\phi$.

2.2 Euclidean model operator

For the heat-kernel analysis we use the Euclidean operator, acting on 4-component spinors on a closed spin manifold,

$$\mathcal{D} = i\gamma^\mu(\nabla_\mu + \kappa a_\mu\gamma_5) + m\gamma_5, \quad (2)$$

where ∇ is the Levi-Civita spin connection, a_μ is the axial vector and $m(x) = y\phi(x)$ the Yukawa mass; $m\gamma_5$ is the Connes–Chamseddine form of a Dirac mass (it anticommutes with the first-order part, and is related to $m\bar{\psi}\psi$ by a chiral rotation of $\psi, \bar{\psi}$). For real κ, a, m the operator is self-adjoint. The Lorentzian coupling in (1) is a *unitary* axial connection, $i\bar{\psi}\gamma^\sigma(\nabla_\sigma - ic\tilde{a}_\sigma\gamma_5)\psi$ with $c = \frac{3}{2}$ (up to the signature conventions of γ_5), i.e. $\kappa = -ic$ in (2). The heat-kernel coefficients below are polynomials in κ^2 , and we give them for general κ^2 ; the Lorentzian case corresponds to

$$\kappa^2 \rightarrow -\kappa^2, \quad c = \frac{3}{2} \text{ (canonically normalised axial field: } c = \frac{3}{2}g_a). \quad (3)$$

[ASSUMPTION] We have not analysed the Wick rotation of the axial field beyond this identification, which we check on the vector analogue (a unitary vector connection is $-ie v_\mu$ and gives the standard QED coefficient, Section 3).

3 Heat-kernel expansion of $\mathcal{D}^2$

3.1 Laplace form

Proposition 3.1 (Laplace-type form of $\mathcal{D}^2$). *[DERIVED] [SCRIPT] $\mathcal{D}^2 = -(g^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu + E)$ with $\tilde{\nabla}_\mu = \nabla_\mu + W_\mu$,*

$$W_\mu = \kappa a^\nu\gamma_{\mu\nu}\gamma_5, \quad E = -\frac{R}{4} - i\gamma^\mu\nabla_\mu B - B^2 - \nabla_\mu W^\mu - W_\mu W^\mu, \quad B = i\kappa \not{a}\gamma_5 + m\gamma_5, \quad (4)$$

and the curvature of $\tilde{\nabla}$ is $\Omega_{\mu\nu} = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma + \nabla_\mu W_\nu - \nabla_\nu W_\mu + [W_\mu, W_\nu]$. In particular $B^2 = \kappa^2 a^2 + m^2$ and $W_\mu W^\mu = -3\kappa^2 a^2$.

Proof. Write $\mathcal{D} = i\not{X} + B$, with $B = i\kappa\not{a}\gamma_5 + m\gamma_5$. Then $\mathcal{D}^2 = -\not{X}^2 + i(\gamma^\mu B + B\gamma^\mu)\nabla_\mu + i\gamma^\mu(\nabla_\mu B) + B^2$. The Lichnerowicz formula gives $-\not{X}^2 = -\nabla^2 + R/4$. Since γ_5 anticommutes with γ^μ , $\gamma^\mu\gamma^\nu\gamma_5 + \gamma^\nu\gamma_5\gamma^\mu = 2\gamma^{\mu\nu}\gamma_5$ and $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$, so $\gamma^\mu B + B\gamma^\mu = 2i\kappa a_\nu\gamma^{\mu\nu}\gamma_5$, and the first-order term is $-2W^\mu\nabla_\mu$ with W_μ as stated. Completing the square, $-(\nabla + W)^2 = -\nabla^2 - 2W\cdot\nabla - (\nabla\cdot W) - W^2$, gives E . B^2 and W^2 follow from $\gamma_5^2 = 1$ and $\sum_\mu\gamma_{\mu\nu}\gamma_{\mu\sigma} = -3\delta_{\nu\sigma} - 2\gamma_{\nu\sigma}$ (the antisymmetric part drops against $a_\nu a_\sigma$). $\square$

The operator identity was verified in flat space by exact Taylor-jet arithmetic on random fields (script `sp_operator_check.py`: residual 2×10^{-15} for the axial-plus-mass operator and 4×10^{-15} for the vector analogue). [SCRIPT] The effective connection $W_\mu = \kappa a^\nu\gamma_{\mu\nu}\gamma_5$ is of the form of a (totally antisymmetric) torsion contribution to the spin connection, as expected for an axial vector.

3.2 Coefficients

For $\Delta = -(\tilde{\nabla}^2 + E)$ the Seeley–DeWitt coefficients are [STANDARD] [8] (up to total derivatives)

$$a_2 = \text{tr}(E + \frac{R}{6}), \quad a_4 = \frac{1}{360}\text{tr}[60RE + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu} + (5R^2 - 2R_{\mu\nu}^2 + 2R_{\mu\nu\rho\sigma}^2)\not{X}], \quad (5)$$

with $\text{tr}\Delta$ -heat-trace $\text{Tr} e^{-t\Delta} \simeq (4\pi t)^{-2} \sum_n t^{n/2} \int \sqrt{g} a_n$. We evaluated (5) with exact 4×4 matrix algebra for (4) on random values of a_μ , $b_{\mu\nu} = \nabla_\mu a_\nu$, m , $p_\mu = \partial_\mu m$ and of an algebraic curvature tensor (Kulkarni–Nomizu sums), and fitted the integrand to a basis of 19 independent invariants (script `sp_heatkernel.py`); the fit has full rank and residual below 6×10^{-12} . [SCRIPT] Integrating by parts with $\int b_{\mu\nu} b^{\mu\nu} = \int [(\nabla\cdot a)^2 - R_{\mu\nu} a^\mu a^\nu]$, one finds

$$a_2 = -\frac{1}{3}R - 4m^2 + 8\kappa^2 a_\mu a^\mu, \quad (6)$$

$$a_4 = \frac{1}{360}(-7R_{\mu\nu\rho\sigma}^2 - 8R_{\mu\nu}^2 + 5R^2) + 2m^4 + \frac{1}{3}m^2 R + 2(\partial m)^2 - 8\kappa^2 m^2 a_\mu a^\mu - \frac{2}{3}\kappa^2 f_{\mu\nu} f^{\mu\nu}, \quad (7)$$

with $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. There is no a^4 term and no parity-odd term, and no V_μ anywhere.

Proposition 3.2 (Coefficients). [DERIVED] [SCRIPT] (6)–(7) hold, and satisfy the following checks.

1. Gravity. For $\kappa = m = 0$, $a_4 = -\frac{1}{20}C^2 + \frac{11}{360}E_4 + 0 \cdot R^2$ with C^2 the Weyl tensor squared and $E_4 = R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2$; the trace anomaly of a Dirac field, $\langle T \rangle = -a_4/16\pi^2$, is $(cC^2 - aE_4)/16\pi^2$ with $(c, a) = (6/120, 11/360)$, the pair quoted in the monograph [2, 9].
2. Mass. The m -dependence for constant m and $\kappa = 0$ is the Taylor expansion of the shifted counting function $N(k^2 - m^2)$ (Section 5): $-4m^2$ in a_2 , $2m^4 + \frac{1}{3}m^2 R$ in a_4 .
3. Vector benchmark. For the vector coupling $\mathcal{D}_V = i\gamma^\mu(\nabla_\mu - ie v_\mu)$ the same code gives $a_4 \supset \frac{2}{3}e^2 f_{\mu\nu} f^{\mu\nu}$ and no v^2 term in a_2 . With $\partial_t \Gamma^F = -\frac{1}{16\pi^2} \int a_4$ this is $\partial_t e^2 = e^4/(6\pi^2)$, i.e. $\beta_e = e^3/(12\pi^2)$.
4. Axial vs. vector. The f^2 coefficient of the axial coupling, $-\frac{2}{3}\kappa^2$, equals the vector one with $e^2 \rightarrow -\kappa^2$; under (3) it is $+\frac{2}{3}c^2$.

Proof. The coefficients are the output of the script. For item 1, with $C^2 = R_{\mu\nu\rho\sigma}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2$, solve $-7R_{\mu\nu\rho\sigma}^2 - 8R_{\mu\nu}^2 + 5R^2 = \alpha C^2 + \beta E_4 + \gamma R^2$: $\alpha = -18$, $\beta = 11$, $\gamma = 0$. For item 3, β_e is the standard QED result, which the script reproduces from $\frac{1}{12}\text{tr}\Omega^2 + \frac{1}{2}\text{tr}E^2$. $\square$

Remark 3.3 (Consequences for the spectral action). (a) The coefficients contain the axial vector, the Yukawa mass, and the geometry; they contain neither V_μ nor $F_{\mu\nu}$. Whatever cutoff function is used, the fermionic spectral action therefore does not generate F^2 , a mass or any coupling of the Weyl vector V_μ . This is the precise sense in which the construction “cannot reach”

the Weyl gauge sector. (b) The axial vector receives a kinetic term, $-\frac{2}{3}\kappa^2 f^2$ per Dirac fermion, in the same way as a vector coupling. It also receives a non-gauge-invariant term $8\kappa^2 \Lambda^2 a^2$ from a_2 : the heat kernel of $\mathcal{D}^2$ does not respect axial gauge invariance, as expected of a regularisation of a chirally coupled field. Its coefficient is scheme dependent and is not identified with the distortion-sector mass of the monograph. (c) The fermion-loop contribution to the running of the axial coupling follows from item 3 of Proposition 3.2 and (3): $\partial_t(1/g_a^2)|_{\text{ferm}} = -n_f c_g^2/(6\pi^2)$, i.e. $\beta_{g_a}|_{\text{ferm}} = +3n_f g_a^3/(16\pi^2)$ for n_f massless Dirac fermions with $c = \frac{3}{2}g_a \rightarrow \frac{3}{2}$ in the normalisation where $\tilde{a}$ has the kinetic term $-\frac{1}{4}g_a^{-2}f^2$. This is the contribution of one block of the one-loop trace in the continuation (3); it ignores the gravitational and Proca blocks, whose contribution to gauge-coupling running is gauge dependent. [ASSUMPTION]

4 Weyl invariance and the dilaton as cutoff scale

The spectral action is $S = \text{Tr}f(\mathcal{D}^2/\Lambda^2) \simeq \frac{1}{16\pi^2} \int \sqrt{g} [f_4 \Lambda^4 a_0 + f_2 \Lambda^2 a_2 + f_0 a_4 + \dots]$ with $f_4 = \int_0^\infty u f(u) du$, $f_2 = \int_0^\infty f(u) du$, $f_0 = f(0)$ and $a_0 = 4$ [5]. The coefficients (6)–(7) show a feature that was already visible in the geometric terms: the term $f_2 \Lambda^2 a_2 \ni -\frac{1}{3} \Lambda^2 R - 4m^2 \Lambda^2 + 8\kappa^2 \Lambda^2 a^2$ is a bare Einstein–Hilbert term, a bare axial mass term and (through $m^2 = y^2 \phi^2$) a bare dilaton quartic. With a fixed external Λ these are not Weyl invariant: $\sqrt{g} R$ and $\sqrt{g} a_\mu a^\mu$ have weight $+2$, which is the content of the no-go theorem of the monograph for bare invariants [2]. In QDG, weight $+2$ is compensated by ϕ^2 . We show that this can be done exactly.

Definition 4.1 (Dilaton-dressed Dirac operator). On a compact spin manifold, with $\phi > 0$ smooth, $\mathcal{D}_\phi \equiv \phi^{-1/2} \mathcal{D} \phi^{-1/2}$, where $\mathcal{D}$ is (2) with $m = y\phi$. It is dimensionless.

Proposition 4.2 (Weyl invariance of the dilaton-dressed spectral action). [DERIVED] [SCRIPT] Under $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $\phi \rightarrow \Omega^{-1} \phi$, $a_\mu \rightarrow a_\mu$ (and $\psi \rightarrow \Omega^{-3/2} \psi$) with $\Omega > 0$ smooth,

$$\mathcal{D}_\phi \longrightarrow \Omega^{-2} \mathcal{D}_\phi \Omega^2,$$

so the spectrum of $\mathcal{D}_\phi$ is Weyl invariant and so is $S_\phi = \text{Tr}f(\mathcal{D}_\phi^2/\kappa_0^2)$ for any constant κ_0 . The operator $\mathcal{D}_\phi$ does not depend on V_μ .

Proof. The Dirac operator is conformally covariant, $\nabla_{\Omega^2 g} = \Omega^{-5/2} \nabla_g \Omega^{3/2}$ (checked at a point with the spin connection computed from the Cartan equation, script `sp_weyl_check.py`: weight $3/2$ passes at 9×10^{-16} , $1/2$ and $5/2$ fail). [SCRIPT] The axial term $i\kappa e_a^\mu a_\mu \gamma^a \gamma_5$ has $e_a^\mu \rightarrow \Omega^{-1} e_a^\mu$ and the Yukawa term $y\phi \gamma_5 \rightarrow \Omega^{-1} y\phi \gamma_5$; both multiplicative terms therefore also satisfy $T' = \Omega^{-5/2} T \Omega^{3/2}$. So the bracket $\mathcal{D}$ obeys $\mathcal{D}' = \Omega^{-5/2} \mathcal{D} \Omega^{3/2}$, and $\mathcal{D}'_\phi = (\Omega^{-1} \phi)^{-1/2} \mathcal{D}' (\Omega^{-1} \phi)^{-1/2} = \Omega^{1/2} \phi^{-1/2} \Omega^{-5/2} \mathcal{D} \Omega^{3/2} \Omega^{1/2} \phi^{-1/2} = \Omega^{-2} \mathcal{D}_\phi \Omega^2$, since multiplication operators commute. A similarity transformation by a bounded, boundedly invertible multiplication operator preserves the domain H^1 and the spectrum. $\square$

Proposition 4.3 (Relation to the unitary-gauge metric). [DERIVED] Let $\hat{g} = (\phi/\phi_0)^2 g$ for a constant $\phi_0 > 0$ and let $\mathcal{D}[\hat{g}, a, \hat{m}]$ be (2) built from $\hat{g}$, the same a_μ and the constant mass $\hat{m} = y\phi_0$. Then $\mathcal{D}[\hat{g}, a, \hat{m}] = \phi_0 U \mathcal{D}_\phi U^{-1}$ with $U = \phi^{-2}$, hence

$$\text{Tr}f\left(\frac{\mathcal{D}_\phi^2}{\kappa_0^2}\right) = \text{Tr}f\left(\frac{\mathcal{D}[\hat{g}, a, \hat{m}]^2}{(\kappa_0 \phi_0)^2}\right),$$

the ordinary spectral action of $\hat{g}$ with cutoff scale $\Lambda = \kappa_0 \phi_0$.

Proof. With $\Omega = \phi/\phi_0$, the conformal covariance used above gives $\mathcal{D}[\hat{g}, a, \hat{m}] = \Omega^{-5/2} \mathcal{D} \Omega^{3/2} = \phi_0 \phi^{-5/2} \mathcal{D} \phi^{3/2}$, and $\phi^{-2}(\phi^{-1/2} \mathcal{D} \phi^{-1/2}) \phi^2 = \phi^{-5/2} \mathcal{D} \phi^{3/2}$. $\square$

QDG form of the expansion. Insert $\hat{g}$ in (6)–(7). With $\sqrt{\hat{g}} = (\phi/\phi_0)^4 \sqrt{g}$, $\hat{R} = (\phi_0/\phi)^2 [R - 6\Box \ln \phi - 6(\partial \ln \phi)^2]$ and $\hat{m} = y\phi_0$, every term of the expansion becomes a Weyl-invariant local functional of g, ϕ, a with ϕ_0 dropping out:

$$\begin{aligned} \sqrt{\hat{g}} \Lambda^4 &\rightarrow \kappa_0^4 \sqrt{g} \phi^4, & \sqrt{\hat{g}} \Lambda^2 \hat{R} &\rightarrow \kappa_0^2 \sqrt{g} \phi^2 \tilde{R}|_{V_\mu = \partial_\mu \ln \phi}, \\ \sqrt{\hat{g}} \Lambda^2 \hat{m}^2 &\rightarrow \kappa_0^2 y^2 \sqrt{g} \phi^4, & \sqrt{\hat{g}} \Lambda^2 \hat{g}^{\mu\nu} a_\mu a_\nu &\rightarrow \kappa_0^2 \sqrt{g} \phi^2 a_\mu a^\mu, \end{aligned} \quad (8)$$

where $\tilde{R} = R - 6\nabla \cdot V - 6V^2$ is the Weyl-invariant curvature of the monograph and the first-order term reproduces $\tilde{R}$ evaluated at the pure-gauge value $V_\mu = \partial_\mu \ln \phi$. Thus ϕ^4 , $\phi^2 \tilde{R}$ and $\phi^2 a^2$ (the last being the axial component of the distortion term $\phi^2 \sum \beta_i I_i[\tilde{D}]$) are produced with the structure of the QDG ultraviolet action. The a_4 terms m^4 , $m^2 R$, f^2 , C^2 , E_4 are of weight zero or are completed in the same way ($\hat{m}^2 \hat{R} \rightarrow y^2 \phi^2 \tilde{R}$, $\hat{m}^4 \rightarrow y^4 \phi^4$). The dilaton-dressed construction is the QDG instance of the scale-invariant spectral action, in which the arbitrary scale is made dynamical by a dilaton [6].

Remark 4.4 (What this does and does not give). (a) The resolution of the bare- R problem is exact and does not require a separate argument. (b) The functional depends on V_μ only through its pure-gauge value: in unitary gauge it is the spectral action of the Einstein metric, and the gauge-invariant massive vector $\hat{V}_\mu = V_\mu - \partial_\mu \ln \phi$ (whose kinetic and mass terms define the Weyl sector of the infrared theory) does not appear. The tension between the construction and the Weyl gauge sector is therefore not an artefact of a fixed Λ ; it is the statement that the fermionic Dirac operator is blind to V_μ (Remark 3.3). (c) The map to $\hat{g}$ is singular where $\phi \rightarrow 0$, the locus studied in the companion note on the Einstein-frame accessibility of the conformal point [3]. (d) Coefficients such as $\kappa_0^2 f_2$ are free; no value is predicted, and none is identified with a coupling of QDG. [OPEN]

5 The Wetterich side

For a quadratic fermionic action $\bar{\psi} \mathcal{D} \psi$ in a bosonic background, with vanishing fermionic mean field, the fermionic block of $\Gamma_k^{(2)}$ is $\mathcal{D}$ and the Wetterich equation [7] gives, with the sign of a closed fermion loop,

$$\partial_t \Gamma_k^F[g, a, \phi] = -\text{Tr} \left[(\mathcal{D} + R_k^F)^{-1} \partial_t R_k^F \right]. \quad (9)$$

This takes the form $\frac{1}{2} \text{STr}$ with the factor 2 from ψ and $\bar{\psi}$ cancelling the $\frac{1}{2}$.

Proposition 5.1 (The fermion block is a spectral counting function). *[DERIVED] Let $\mathcal{D}$ be self-adjoint and $R_k^F = \mathcal{D}(k/|\mathcal{D}| - 1)\theta(k^2 - \mathcal{D}^2)$ (the Litim-type fermionic cutoff, a function of $\mathcal{D}$ itself). Then*

$$\partial_t \Gamma_k^F = -\text{Tr} \theta(k^2 - \mathcal{D}^2) \equiv -N_{\mathcal{D}}(k), \quad (10)$$

the number of eigenvalues of $\mathcal{D}^2$ below k^2 . In the same way, for a Laplace-type operator Δ with the Litim cutoff $R_k = (k^2 - \Delta)\theta(k^2 - \Delta)$, $\frac{1}{2} \text{Tr}[\partial_t R_k(\Delta + R_k)^{-1}] = N_{\Delta}(k)$.

Proof. On an eigenspace with eigenvalue λ , $|\lambda| < k$: $\lambda + R_k = k \text{sgn} \lambda$ and $\partial_t R_k = \lambda k/|\lambda| = k \text{sgn} \lambda$, so the ratio is 1; for $|\lambda| > k$, $R_k = 0 = \partial_t R_k$ (the boundary term vanishes since $k/|\lambda| - 1 = 0$ at $|\lambda| = k$). The bosonic case is the same computation with $2k^2\theta/k^2 = 2\theta$. $\square$ $\square$

With the large- k expansion of the counting function, $N_{\Delta}(k) \simeq \frac{1}{16\pi^2} \int \sqrt{g} [\frac{1}{2} k^4 a_0 + k^2 a_2 + a_4 + \dots]$ (the moments of $\theta(1-u)$ are $f_4 = \frac{1}{2}$, $f_2 = 1$, $f_0 = 1$), the fermion block gives from (6)–(7)

$$\partial_t \Gamma_k^F \simeq -\frac{1}{16\pi^2} \int \sqrt{g} \left[2k^4 + k^2 \left(-\frac{1}{3} R - 4m^2 + 8\kappa^2 a^2 \right) + a_4 \right]. \quad (11)$$

Matching with the Einstein–Hilbert flow. Writing $\Gamma \ni \frac{1}{16\pi G} \int \sqrt{g}(-R + 2\bar{\Lambda})$, $G = g/k^2$, $\bar{\Lambda} = \lambda k^2$ and $\partial_t(1/G) = -\eta/G$, the k^4 and $k^2 R$ terms of $\partial_t \Gamma$ give a contribution $g X/(6\pi)$ to η and $gY/(4\pi)$ to the volume term of $\beta_\lambda = -(2 - \eta)\lambda + \dots$, with $X = 6c_R$, $Y = 2c_{\text{vol}}$ where $c_{\text{vol}}k^4 + c_R k^2 R$ is the coefficient in units of $1/16\pi^2$. This is the form used in the monograph [2] (Litim cutoff, $\Phi_1^1(0) = 1$, $\Phi_2^1(0) = \frac{1}{2}$) for matter whose cutoff is a function of the full fluctuation operator (“type B”). Applying Proposition 5.1 to the minimal-matter operators (script `sp_wetterich_matching.py`) [SCRIPT]:

Table 1: Counting-function coefficients of the minimal-matter operators (c_{vol}, c_R in units of $1/16\pi^2$) and the resulting (X, Y) , which coincide with the values of the monograph. The vector is in Feynman gauge with two scalar ghosts; Δ_1 is the Hodge–de Rham Laplacian ($E = -\text{Ric}$). The Dirac fermion is $\mathcal{D}^2 = -\nabla^2 + R/4$.

field	operator(s)	c_{vol}	c_R	X	Y
real scalar	Δ_0	$\frac{1}{2}$	$\frac{1}{6}$	1	1
Dirac fermion	$\mathcal{D}^2$	-2	$\frac{1}{3}$	2	-4
massless vector	$\Delta_1 - 2\Delta_0$	1	$-\frac{2}{3}$	-4	2

For n_f Dirac fermions and two Proca fields in the massless limit (each a vector plus one scalar) this gives $X = 2n_f - 6$ and $Y = 6 - 4n_f$, the QDG-like content used in the companion papers on the Einstein–Hilbert flow. The shift rule of Proposition 3.2(2) and the matching of η and β_λ from the table were checked by the script. [SCRIPT] The relation (10) is the precise sense in which the Wetterich trace of the fermion block is a spectral action, with the test function $f = \theta$, and (11) is its heat-kernel expansion.

Remark 5.2 (Scope of the identification). (a) The identification is exact for each block with a regulator that is a function of that block’s operator; with a regulator that depends on $-\nabla^2$ only (“type A”) the endomorphism E is not absorbed and the coefficients X differ (the monograph gives $X_D = X_V = -1$); the identification of the Wetterich trace with a spectral action is therefore scheme dependent. (b) The graviton block, with its gauge fixing and the non-minimal operators of the Hessian, is not of this form with the physical background, and the Proca mass thresholds are not part of (10) either; a single operator (a spectral triple) whose spectral action reproduces the full trace is not known. (c) The coefficient of f^2 in (11) is the scheme-independent logarithmic one, i.e. the contribution to β_{g_a} in Remark 3.3(c); the coefficients of k^4 , k^2 are scheme dependent.

6 What is and is not established

Statement	Status
The fermion couples to the axial vector with strength $\frac{3}{2}$, not to V_μ	derived in the monograph; re-run
$\mathcal{D}^2$ is of Laplace type, with W_μ and E of (4)	derived, script
a_2, a_4 of (6)–(7); (c, a) , QED and mass checks	script
No coefficient contains V_μ	derived
Fermion-loop $\beta_{g_a} = +3n_f g_a^3/16\pi^2$	continuation assumed
Dilaton-dressed spectral action is Weyl invariant; equals the spectral action of $\hat{g}$	derived
Fermion block of the Wetterich flow $= -N_{\mathcal{D}}(k); (X, Y)$ of Table 1	derived, script
Spectral origin of $\hat{V}_\mu$ kinetic/mass terms; of the graviton block; of Proca masses	open
Wick rotation of the axial coupling, self-adjoint model vs. unitary theory	open
Whether the Weyl-gauged sector changes the flow	open (as in the companion papers)

Open Problem 6.1 (Spectral origin of the Weyl vector sector). [OPEN] If V_μ does not enter the Dirac operator, can the kinetic and mass terms of the gauge-invariant vector $\hat{V}_\mu$ come from another part of a spectral triple (a bosonic operator built from the dilaton and the connection), or are they not spectral quantities in QDG?

Open Problem 6.2 (Physical continuation of the axial coupling). [OPEN] The Euclidean operator (2) is self-adjoint for real κ , whereas the continuation of the Lorentzian Hermitian coupling has $\kappa^2 < 0$ and is not self-adjoint. Define the spectral action (and the fermionic regulator of Proposition 5.1) for the latter, and verify (3) beyond the analogy with the vector coupling.

Open Problem 6.3 (Bosonic sector as spectral functions). [OPEN] Express the graviton, ghost and Proca blocks of the Wetterich trace in the Einstein–Hilbert truncation as spectral counting functions of operators on $\hat{g}$, and decide whether they are the spectrum of one operator.

Open Problem 6.4 (Spectral versus Wetterich couplings). [OPEN] Compare the couplings induced by the dilaton-dressed spectral action at the scale $\kappa_0\phi_0$ with the couplings run by the Wetterich equation, including the free constants $f_{4,2,0}$ and κ_0 .

7 Conclusions

The Dirac operator of QDG contains the metric, the dilaton and the axial vector, and does not contain the Weyl vector. We computed the Seeley–DeWitt coefficients of its square, including the axial and Yukawa couplings, and checked them against exact operator identities, the QED beta function, the Dirac conformal-anomaly pair, and the mass-shift rule of the counting function. The spectral action built from this operator cannot generate any term involving V_μ . With the dilaton as cutoff scale, it is exactly Weyl invariant, equals the ordinary spectral action of the unitary-gauge metric, and has the structure of the QDG ultraviolet action evaluated at the pure-gauge value of V_μ . For the fermion block, and for scalar and vector blocks with regulators adapted to their operators, the Wetterich trace equals a spectral counting function, which reproduces the matter coefficients (X, Y) of the Einstein–Hilbert flow. The Weyl vector sector, the graviton block and the Proca masses lie outside this correspondence; whether they can be brought inside it is open. Nothing here bears on whether the flow of QDG has a fixed point.

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