

The WISC Criterion in the Einstein Frame: Accessibility of the Conformal Point in Scalar-Tensor Gravity

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Abstract

The WISC (Weyl-Invariant Scalar Coupling) question is whether the point $\phi = 0$, where the effective Planck mass $\phi^2/6$ of the scalar-tensor theory $\frac{1}{12}\phi^2 R - \frac{1}{2}(\partial\phi)^2 - V(\phi)$ vanishes, can be reached dynamically. A naive rule, “the theory is singular if and only if $V(0) = 0$ ”, is shown to fail in both directions. We reduce the cosmological problem exactly to a canonical scalar field in the Einstein frame, $\chi = \sqrt{12} M \ln \phi$ with potential $\tilde{V} = 36M^4 V/\phi^4$, in which $\phi \rightarrow 0$ is $\chi \rightarrow -\infty$. The fate of $\phi = 0$ is controlled by the sign of $w'(\phi)$, where $w = V/\phi^4$ measures the explicit breaking of the global scale symmetry. *For $V > 0$ we prove that $\phi = 0$ is inaccessible if $w' \leq 0$ near the origin, and is approached monotonically, only as $t \rightarrow \infty$ (Type II), from an explicit open set of initial data if $w' > 0$.* For $V < 0$ near the origin, numerical integrations give finite-time crunches (Type I) with the analytic arrival law $\phi \propto (t_s - t)^{1/(3\sqrt{2}-1)}$. The reduction is checked against a direct integration of the Jordan-frame equations to 2×10^{-12} . In Quantum Distortion Gravity the dilaton is pure gauge, so accessibility of $\phi = 0$ is a statement about a singular gauge transformation; the only Weyl-invariant potential, $V = \lambda\phi^4$, is regular. The Weyl-gauged sector beyond the dilaton is not analysed.

Keywords: quantum gravity, metric-affine gravity, Weyl symmetry, scalar-tensor gravity, Einstein frame, cosmological singularity

1 Introduction

Scalar-tensor theories in which a scalar ϕ generates the Planck mass through a non-minimal coupling $\phi^2 R/12$ contain a special point, $\phi = 0$, where the effective Planck mass $M_{\text{eff}}^2 = \phi^2/6$ vanishes. Whether this point is dynamically accessible decides whether the Jordan-frame description breaks down. The question is of interest for Quantum Distortion Gravity (QDG) [1, 2], whose dilaton has exactly this coupling, for α -attractor models, whose conformal formulation has a similar pole structure [3], and for the literature on Weyl-invariant cosmology and the passage through $\phi = 0$ into an “antigravity” region [4, 5].

Within the QDG programme a working rule has been used for a classification of such potentials by the WISC criterion¹: the theory is *WISC-singular* if and only if $V(\phi) \rightarrow 0$ as $\phi \rightarrow 0$. We show that this rule is incomplete: the mass term $\frac{1}{2}m^2\phi^2$ and the quartic $\lambda\phi^4$ have $V(0) = 0$ yet $\phi = 0$ is inaccessible, while a constant $V(0) < 0$ is not regular. The contributions of this note are:

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¹The rule was formulated as a working tool in the QDG programme and has no separate publication.

1. the Friedmann constraint of the theory, derived from the minisuperspace action (Sec. 2), and the observation that the relation $\dot{\phi}^2|_{\phi=0} = -2V(0)$ requires $H\phi \rightarrow 0$;
2. an exact reduction to a canonical scalar field in the Einstein frame (Sec. 3), which turns the question into the behaviour of $\tilde{V}$ at $\chi \rightarrow -\infty$;
3. a *corrected WISC criterion*, stated and proved for $V > 0$ (Theorem 4.3), together with numerically supported statements for $V < 0$, including the arrival law at a Type I singularity (Sec. 4);
4. numerical illustrations with scripts (Sec. 5), and the consequences for QDG, where the dilaton is pure gauge (Sec. 6).

Throughout we use signature $(-, +, +, +)$, $\hbar = c = 1$, and a reference mass M (set to 1 in numerical illustrations). Logical status is tagged [DERIVED], [NUMERICAL], [OPEN]. All numbers come from the scripts in `code/` (numpy only).

2 Framework

2.1 Action and Brans–Dicke interpretation

We consider

$$S = \int d^4x \sqrt{-g} \left[\frac{\phi^2}{12} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right] + S_m. \quad (1)$$

With $\Phi = \phi^2/12$ this is Brans–Dicke theory with parameter $\omega_{\text{BD}} = 3/2$. Action (1) is not locally Weyl invariant: with a standard-sign kinetic term the locally Weyl-invariant scalar carries $-\frac{1}{12}R\phi^2$, i.e. $\omega_{\text{BD}} = -3/2$. It is invariant under the *global* rescaling $g \rightarrow \Omega^2 g$, $\phi \rightarrow \Omega^{-1}\phi$ (Ω constant) if $V = \lambda\phi^4$. Local invariance in QDG is obtained differently, by gauging with a Weyl vector V_μ (Sec. 6).

2.2 Field equations

For $ds^2 = -N^2 dt^2 + a^2 d\vec{x}^2$ the minisuperspace Lagrangian is $L = Na^3 [\frac{F}{2}R + \frac{\dot{\phi}^2}{2N^2} - V]$ with $F = \phi^2/6$ and $R = 6(\frac{\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}\dot{N}}{aN^3})$. Using

$$a^2\ddot{a}/N - a^2\dot{a}\dot{N}/N^2 = \frac{d}{dt}(a^2\dot{a}/N) - 2a\dot{a}^2/N$$

and integrating by parts, with $\dot{F} = \phi\dot{\phi}/3$,

$$L = -\frac{\phi\dot{\phi}a^2\dot{a}}{N} - \frac{\phi^2}{2} \frac{a\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N} - Na^3V(\phi). \quad (2)$$

The lapse equation $\partial L/\partial N = 0$ at $N = 1$ gives, after division by $a^3/2$,

$$\boxed{\phi^2 H^2 + 2H\phi\dot{\phi} - \dot{\phi}^2 = 2V(\phi)} \quad (3)$$

(in the presence of minimally coupled matter, add $2\rho_m$ to the right-hand side). The Euler–Lagrange equation for ϕ , using $\ddot{a}/a = \dot{H} + H^2$, is

$$\ddot{\phi} + 3H\dot{\phi} - \phi(2H^2 + \dot{H}) + V'(\phi) = 0. \quad (4)$$

[DERIVED] Equations (3) and (4) determine the dynamics. The time derivative of (3) and Eq. (4) are two linear equations for $(\dot{\phi}, \dot{H})$ with determinant $4\phi^2 H$; the Jordan-frame system is therefore degenerate at $\phi = 0$ and at $H = 0$, which is one more reason to prefer the Einstein frame below. The Jordan-frame integrations of this note use exactly this system and avoid $H = 0$.

2.3 What the constraint says at $\phi = 0$

Let $x \equiv \lim H\phi$ along a trajectory reaching $\phi = 0$. Equation (3) gives

$$x^2 + 2x\dot{\phi} - \dot{\phi}^2 = 2V(0). \quad (5)$$

If $x = 0$ (i.e. H grows more slowly than $1/\phi$) this reduces to $\dot{\phi}^2 = -2V(0)$, which would forbid arrival for $V(0) > 0$. But for $x \neq 0$ Eq. (5) has real solutions $x = -\dot{\phi} \pm \sqrt{2\dot{\phi}^2 + 2V(0)}$ for any $V(0) > -\dot{\phi}^2$, in particular for $V(0) > 0$. A kinematic argument at $\phi = 0$ is therefore not a proof of inaccessibility; accessibility has to be decided dynamically, which is done in Sec. 3.

3 Exact reduction to the Einstein frame

Proposition 3.1 (Einstein-frame map). *[DERIVED] Define $\tilde{g}_{\mu\nu} = (F/M^2) g_{\mu\nu}$ with $F = \phi^2/6$, and*

$$\chi = \sqrt{12} M \ln(\phi/\phi_*), \quad \tilde{V}(\chi) = \frac{M^4 V(\phi)}{F^2} = \frac{36 M^4 V(\phi)}{\phi^4}. \quad (6)$$

Then (1) (without extra matter) is equivalent to $\int \sqrt{-\tilde{g}} [\frac{M^2}{2} \tilde{R} - \frac{1}{2}(\tilde{\partial}\chi)^2 - \tilde{V}(\chi)]$. In flat FLRW, with $d\tau = (\phi/\sqrt{6}M) dt$ and $\tilde{a} = \phi a/(\sqrt{6}M)$,

$$3M^2 \tilde{H}^2 = \frac{1}{2} \chi'^2 + \tilde{V}, \quad \chi'' + 3\tilde{H}\chi' + \tilde{V}_{,\chi} = 0, \quad \tilde{H}' = -\frac{\chi'^2}{2M^2}, \quad H = \frac{\phi}{\sqrt{6}M} \tilde{H} - \frac{\dot{\phi}}{\phi}, \quad (7)$$

where a prime is $d/d\tau$.

Proof. For $\mathcal{L} = \frac{F}{2} R - \frac{1}{2}(\partial\phi)^2 - V$ the standard conformal transformation gives the Einstein-frame kinetic term $\frac{1}{2}[M^2/F + \frac{3}{2}M^2(F'/F)^2](\partial\phi)^2$ (here $F' = dF/d\phi$). With $F = \phi^2/6$, $F'/F = 2/\phi$, the bracket is $6M^2/\phi^2 + 6M^2/\phi^2 = 12M^2/\phi^2$, so $d\chi = \sqrt{12}M d\phi/\phi$. The potential rescales as $V \rightarrow M^4 V/F^2$. The relation for H follows from $\tilde{a} = \sqrt{F}a/M$ and $d\tau = \sqrt{F}dt/M$: $\tilde{H} = (M/\sqrt{F})(H + \dot{F}/2F)$, with $\dot{F}/F = 2\dot{\phi}/\phi$. The equation $\tilde{H}' = -\chi'^2/(2M^2)$ follows by differentiating the first equation and using the second. $\square$

Proposition 3.2 (Equivalence of the constraints). *[DERIVED] Under $H = \phi\tilde{H}/(\sqrt{6}M) - \dot{\phi}/\phi$ and $\chi' = 6\sqrt{2}M^2\dot{\phi}/\phi^2$, the Jordan constraint (3) is the Einstein-frame Friedmann equation $3M^2\tilde{H}^2 = \frac{1}{2}\chi'^2 + \tilde{V}$.*

Proof. Substituting, $\phi H + \dot{\phi} = \phi^2 \tilde{H}/(\sqrt{6}M)$, and $\frac{1}{2}\chi'^2 + \tilde{V} = 36M^4(\dot{\phi}^2 + V)/\phi^4$, so the Friedmann equation reads $(\phi H + \dot{\phi})^2 = 2\dot{\phi}^2 + 2V$, which is (3) after expanding the square. $\square$

Remark 3.3 (Numerical check). *[NUMERICAL] Script `wisc_frames.py` verifies the algebraic equivalence on 2000 random points (relative residual 6×10^{-16}) and integrates the Jordan-frame system ((4) together with the time derivative of (3), solved for $\ddot{\phi}$ and $\dot{H}$) and the Einstein-frame system (7), both in Jordan time and from the same initial data, for five potentials up to $t = 4$. The two integrations agree in $\phi(t)$ to 1.5×10^{-12} and in $H(t)$ to 8×10^{-13} , and the Jordan constraint drifts by at most 3×10^{-13} .*

Remark 3.4 (The divergence of H at $\phi \rightarrow 0$ is a frame effect). In (7), $H = \frac{\phi}{\sqrt{6}M} \tilde{H} - \dot{\phi}/\phi$. As $\phi \rightarrow 0$ with $\dot{\phi} \neq 0$ the second term diverges even if $\tilde{H}$ is finite. This reflects $a = \sqrt{6}M\tilde{a}/\phi$, i.e. a conformal factor, and is not by itself a curvature singularity of the Einstein-frame geometry. Conversely, $\phi \rightarrow 0$ corresponds to $\chi \rightarrow -\infty$, an infinite field distance.

4 The corrected WISC criterion

4.1 Definitions

Definition 4.1 (Accessibility and WISC classes). Consider expanding solutions ($\tilde{H} > 0$ initially) of the vacuum system. The point $\phi = 0$ is *accessible* if there is an open set of initial data $(\phi_i, \dot{\phi}_i)$, $\phi_i > 0$, whose solutions satisfy $\phi(t) \rightarrow 0$. The theory is *WISC-regular* if $\phi = 0$ is not accessible; it is *WISC-singular of Type I* if arrival occurs at a finite Jordan time t_s , and *of Type II* if it occurs only as $t \rightarrow \infty$.

Definition 4.2 (Scale-breaking function). $w(\phi) \equiv V(\phi)/\phi^4$.

By (6), $\tilde{V} = 36M^4 w(\phi)$. Hence w is constant if and only if $V = \lambda\phi^4$, the potential that preserves the global scale symmetry of (1); w' measures the explicit breaking of that symmetry. Since $d\chi = \sqrt{12}M d\phi/\phi$,

$$\tilde{V}_{,\chi} = \frac{36M^3}{\sqrt{12}} \phi w'(\phi), \quad \text{sign } \tilde{V}_{,\chi} = \text{sign } w'. \quad (8)$$

With $E = \frac{1}{2}\chi'^2 + \tilde{V}$, Eq. (7) gives $E' = -3\tilde{H}\chi'^2$.

4.2 The criterion for positive potentials

Theorem 4.3 (Corrected WISC criterion, $V > 0$). *[DERIVED] Let $V(\phi) > 0$ for all $\phi > 0$, and suppose w' has a constant sign on $(0, \phi_0)$ for some $\phi_0 > 0$. Then:*

- (a) *if $w' \leq 0$ on $(0, \phi_0)$, every expanding solution satisfies $\phi \geq \phi_{\min} > 0$, with $\phi_{\min}$ depending on the initial data: the theory is **WISC-regular**;*
- (b) *if $w' > 0$ on $(0, \phi_0)$, every expanding solution that at some time has $\phi < \phi_0$ and $\dot{\phi} \leq 0$ satisfies $\phi(t) \downarrow 0$ monotonically, with $t \rightarrow \infty$: the theory is **WISC-singular of Type II**, with inner basin $\{\phi < \phi_0, \dot{\phi} \leq 0\}$.*

In compact form, within the class of potentials with $V > 0$ and eventually constant sign of w' ,

$$\text{WISC-regular} \iff \frac{d}{d\phi} \left(\frac{V}{\phi^4} \right) \leq 0 \text{ near } \phi = 0, \quad \text{Type II} \iff \frac{d}{d\phi} \left(\frac{V}{\phi^4} \right) > 0 \text{ near } \phi = 0.$$

(9)

Proof. Since $V > 0$, $\tilde{V} > 0$ and $3M^2\tilde{H}^2 = \frac{1}{2}\chi'^2 + \tilde{V} > 0$: $\tilde{H}$ never vanishes, so the universe expands forever, and $E' = -3\tilde{H}\chi'^2 \leq 0$. Hence $|\chi'| \leq \sqrt{2E_i}$ at all times. Write $\chi_0 = \sqrt{12}M \ln(\phi_0/\phi_*)$.

(a) By (8), $\tilde{V}_{,\chi} \leq 0$ for $\chi < \chi_0$, so $\tilde{V}$ is non-increasing in χ and $\tilde{V}(\chi) \geq \tilde{V}_0 \equiv \tilde{V}(\chi_0) > 0$ there; therefore $\tilde{H} \geq H_0 \equiv \sqrt{\tilde{V}_0/3}/M > 0$ in this region. In the region, $\chi'' = -3\tilde{H}\chi' - \tilde{V}_{,\chi} \geq -3\tilde{H}\chi'$, so at a point with $\chi' = 0$ one has $\chi'' \geq 0$ and χ' cannot change from positive to negative. A descent ($\chi' < 0$) in the region therefore either is the initial one or begins at the entry point $\chi = \chi_0$, with $|\chi'_e| \leq \sqrt{2E_i}$. During a descent,

$$\chi'' = -3\tilde{H}\chi' - \tilde{V}_{,\chi} \geq 3H_0|\chi'|, \quad \text{so} \quad \frac{d|\chi'|}{d\tau} \leq -3H_0|\chi'|,$$

and the excursion is at most $|\chi'_e|/(3H_0) \leq \sqrt{2E_i}/(3H_0)$. Thus $\chi \geq \min(\chi_i, \chi_0) - \sqrt{2E_i}/(3H_0) > -\infty$, i.e. $\phi \geq \phi_{\min} > 0$.

(b) Now $\tilde{V}_{,\chi} > 0$ for $\chi < \chi_0$. Suppose $\chi < \chi_0$ and $\chi' \leq 0$ at τ_0 . At any time in the region where $\chi' = 0$ one has $\chi'' = -\tilde{V}_{,\chi} < 0$, so χ' cannot become positive: χ decreases monotonically

and stays in the region. Suppose $\chi \rightarrow \chi_\infty > -\infty$. Then $\tilde{V}_{,\chi}(\chi(\tau)) \rightarrow \tilde{V}_{,\chi}(\chi_\infty) \equiv 2c > 0$, since V is smooth at $\phi_\infty > 0$. Because $3M^2\tilde{H}^2 = E \leq E_i$, $\tilde{H} \leq H_{\max} \equiv \sqrt{E_i/3}/M$, so for large τ , $\chi'' = 3\tilde{H}|\chi'| - \tilde{V}_{,\chi} \leq 3H_{\max}|\chi'| - c$. Whenever $|\chi'| < c/(6H_{\max})$ this gives $\chi'' \leq -c/2$, so $|\chi'|$ grows at rate at least $c/2$ until it exceeds $c/(6H_{\max})$, and it cannot fall back below that value. Then $\int |\chi'| d\tau = \infty$, contradicting convergence of χ . Hence $\chi \rightarrow -\infty$, i.e. $\phi \rightarrow 0$. Because $|\chi'| \leq \sqrt{2E_i}$, this takes infinite τ , and $t = \int \sqrt{6}M d\tau/\phi \geq \sqrt{6}M(\tau - \tau_0)/\phi_0 \rightarrow \infty$. $\square$

Corollary 4.4 (Power laws and standard cases). *[DERIVED] For $V = \lambda\phi^n$ with $\lambda > 0$: WISC-regular if $n \leq 4$, WISC-singular of Type II if $n > 4$. Potentials with $V(0) > 0$ are regular. For $V = V_0e^{-1/\phi}$, $w'/w = (1 - 4\phi)/\phi^2$, so $w' > 0$ exactly for $\phi < 1/4$: Type II with inner basin $\phi < 1/4$. For $V = \lambda\phi^4(1 + b\phi^s)$, $b > 0$: Type II; the slow-roll approach gives $\phi \propto \tau^{-1/s}$.*

Remark 4.5 (Consequences for the naive rule). The naive rule (“singular if and only if $V \rightarrow 0$ ”) fails in both directions. It calls the mass term $\frac{1}{2}m^2\phi^2$, the quartic $\lambda\phi^4$ and $\lambda\sqrt{\phi}$ singular, whereas by the Corollary they are regular ($\tilde{V} \propto \phi^{-2}$, const, $\phi^{-7/2}$). Conversely, a constant $V(0) < 0$ is not regular (Sec. 4.3).

Remark 4.6 (The slow-roll law). *[NUMERICAL] For $V = \lambda\phi^4(1 + b\phi^s)$ one has $\tilde{V}_\infty = 36M^4\lambda$, $\tilde{H} \simeq \sqrt{12\lambda}M$ and $\tilde{V}_{,\chi} = \frac{36M^3}{\sqrt{12}}\lambda bs\phi^s$. The slow-roll equation $\chi' \simeq -\tilde{V}_{,\chi}/(3\tilde{H})$ gives $\phi'/\phi = -\lambda bs\phi^s/(\sqrt{12\lambda}M)$, hence $\phi^{-s} \simeq (\lambda bs^2/\tilde{H})\tau$. For $\lambda = 0.01$, $b = 1/2$, $s = 2$ ($M = 1$) the prediction is $\phi \simeq 4.16\tau^{-1/2}$, i.e. 0.131 at $\tau = 10^3$ and 0.0416 at $\tau = 10^4$; the integration (`wisc_table.py`) gives 0.1307 and 0.04161. The Jordan time grows as $\tau^{1+1/s}$, so $\phi \propto t^{-1/(s+1)}$.*

4.3 Negative potentials near the origin

If $V < 0$ near $\phi = 0$ then $\tilde{V} < 0$ there, $\tilde{H}$ can vanish, and the Einstein-frame universe can recollapse; Theorem 4.3 does not apply, because E is no longer monotone after recollapse.

Numerical statement. *[NUMERICAL] In every case integrated with $V < 0$ near the origin ($V(0) < 0$; $-\lambda\phi^2$; $-\lambda\phi^4$; $-\lambda\phi^n$, $n = 6, 8, 12, 13$; Table 2) the solution recollapses and $\phi \rightarrow 0$ at a finite Jordan time (Type I) on an open set of initial data.*

Recollapse threshold for $V = -\lambda\phi^n$. In pure kination, $\chi' = -\frac{\sqrt{6}M}{3\tau}$, $\chi = -\frac{\sqrt{6}M}{3}\ln\tau$, so $\phi \propto \tau^{-1/(3\sqrt{2})}$, the potential tail decays as $|\tilde{V}| \propto \tau^{-(n-4)/(3\sqrt{2})}$ and the kinetic energy as τ^{-2} . The tail wins asymptotically, so recollapse is expected, iff $(n-4)/(3\sqrt{2}) < 2$, i.e. $n < 4 + \sqrt{72} \approx 12.49$. *[DERIVED]* (kination estimate) The estimate is asymptotic, and near the threshold the time of recollapse depends strongly on the initial data. Script `wisc_threshold.py` (event $E = 0$ in the constraint-eliminated system, $\lambda = 0.01$, up to $\tau = 10^{24}$) gives: for $(\phi_i, \chi'_i) = (1, -2)$, recollapse for $n \leq 13$ (at $\tau = 4 \times 10^3$ for $n = 13$) and none for $n \geq 14$; for $(1, -6)$, recollapse for $n \leq 12.4$ (at $\tau = 10^{15}$ for $n = 12.4$) and none for $n \geq 12.6$; for $(0.5, -3)$, recollapse for $n \leq 12$ (at $\tau = 4 \times 10^{22}$ for $n = 12$) and none for $n \geq 12.4$. Recollapse always occurs for $n \leq 11$ and never for $n \geq 14$, in agreement with the estimate; between 12 and 14 the outcome is initial-data dependent, so a sharp threshold is not established numerically. *[NUMERICAL]*

Remark 4.7 (Direction of the crunch). Near a kinetic-dominated crunch χ diverges logarithmically in the direction of χ' : $\phi \rightarrow 0$ if $\chi' < 0$, $\phi \rightarrow \infty$ if $\chi' > 0$. For $-\lambda\phi^2$ started with $\chi' > 0$ the solution ends in a crunch at large ϕ ($\phi > 20$ and growing at $\tilde{H} = -10^4$, Table 2). Type I therefore refers to the generic infall case $\chi' < 0$.

Remark 4.8 (Arrival law at a Type I singularity). *[DERIVED]* In the kinetic-dominated crunch, $\tilde{H} = -1/[3(\tau_c - \tau)]$, $\tilde{a} \propto (\tau_c - \tau)^{1/3}$ and $\chi = \frac{\sqrt{6}M}{3}\ln(\tau_c - \tau)$, so $\phi \propto (\tau_c - \tau)^q$ with $q = 1/(3\sqrt{2})$. With $dt = \sqrt{6}M d\tau/\phi$ one finds $t_s - t \propto (\tau_c - \tau)^{1-q}$, hence

$$\phi \propto (t_s - t)^p, \quad p = \frac{q}{1-q} = \frac{1}{3\sqrt{2}-1} \approx 0.308, \quad (10)$$

with $\dot{\phi} \rightarrow \infty$. The Jordan Hubble rate is $H = \frac{\dot{\phi}}{\sqrt{6}M}(\tilde{H} - \chi'/(\sqrt{12}M))$ (Eq. (7) with $\dot{\phi}/\phi = \phi\chi'/(6\sqrt{2}M^2)$), which gives $H \simeq -s/(t_s - t)$ and

$$a_J \propto (t_s - t)^s, \quad s = \frac{\sqrt{2} - 1}{3\sqrt{2} - 1} \approx 0.128, \quad (11)$$

so the Jordan-frame universe is contracting ($a_J \rightarrow 0$). [NUMERICAL] Script `wisc_arrival.py` (case $-0.01\phi^2$, data $(1, -1.5)$, $\tilde{H}$ down to -10^7) gives the local slope $d \ln \phi / d \ln(t_s - t) = 0.3084$ at each of five decades (with t_s estimated from the last point) and $(t_s - t)H = -0.1277$, against the analytic 0.30839 and 0.12774. This is not the linear arrival $\phi \propto t_s - t$ that a naive extension of Eq. (5) would suggest.

4.4 Basins of initial data

Theorem 4.3(b) makes an inner basin explicit: the region where the force points toward $\phi = 0$, i.e. $w' > 0$, bounded by the first zero ϕ_0 of w' , with $\dot{\phi} \leq 0$. For $V_0 e^{-1/\phi}$ the barrier is at $\phi = 1/4$; for $-V_0 + \frac{1}{2}m^2\phi^2$, $w' \propto 4V_0 - m^2\phi^2$ and the barrier is at $\phi_0 = \sqrt{4V_0/m^2} \approx 0.632$ for $V_0 = 0.01$, $m^2 = 0.1$.

The basin is not limited to this inner region. Hubble friction limits the excursion in χ , but only logarithmically in the kinetic energy: in kination $\Delta\chi \simeq \sqrt{6}M \Delta \ln \tilde{a}$ while the kinetic energy falls as $\tilde{a}^{-6}$, so the excursion is about $\frac{\sqrt{6}}{6}M \ln(\frac{1}{2}\chi'^2/\tilde{V})$ and grows without bound with the initial velocity. [NUMERICAL] Script `wisc_basin.py` (started at $\phi_i = 1$ with $\chi' = -\kappa$) shows this: for $V_0 e^{-1/\phi}$ ($V_0 = 0.1$) the field is reflected at $\phi_{\min} = 0.946, 0.623, 0.359, 0.259$ for $\kappa = 1, 16, 256, 10^3$, and crosses the barrier $1/4$ and keeps falling for $\kappa = 3 \times 10^3$ and 10^4 ; for $-V_0 + \frac{1}{2}m^2\phi^2$ it is reflected for $\kappa \leq 8$ ($\phi_{\min} = 0.635$ at $\kappa = 8$, just outside the barrier 0.632) and recollapses after crossing for $\kappa \geq 16$. Thus accessibility holds on an open set of initial data, containing the inner basin, not on all initial data; the full basin depends on the kinetic energy and is not characterised here. [OPEN]

Table 1: The corrected WISC classification. The first two rows are theorems; the third is numerical.

Condition near $\phi = 0$	Examples	Fate of $\phi = 0$	Status
$V > 0, w' \leq 0$	$V(0) > 0; \lambda\phi^n$ ($0 < n \leq 4$)	inaccessible: Regular	Thm. 4.3(a)
$V > 0, w' > 0$	$\lambda\phi^n$ ($n > 4$); $V_0 e^{-1/\phi}$	monotone, $t \rightarrow \infty$: Type II	Thm. 4.3(b)
$V < 0$	$V(0) < 0; -\lambda\phi^2; -\lambda\phi^4$	finite-time crunch: Type I	numerical

5 Numerical illustrations

We integrated the Einstein-frame system (7) ($M = 1$, ϕ in units of M , expanding at $\tau = 0$; adaptive Dormand–Prince, relative tolerance 10^{-12}) from (ϕ_i, χ'_i) for the potentials of Table 2 (script `wisc_table.py`). These are illustrations for the stated parameters, not proofs; the last column indicates whether Theorem 4.3 covers the case. In the cases with crunch the constraint $3\tilde{H}^2 = \frac{1}{2}\chi'^2 + \tilde{V}$, which is not used in the integration, is satisfied to a relative 2×10^{-12} in all crunch cases except $n = 12$ (7×10^{-10}) and $n = 13$ (6×10^{-6}).

Theorem 4.3(a) provides the lower bound $\phi_{\min} \geq \phi_i \exp[-\sqrt{2E_i}/(3H_0\sqrt{12}M)]$ for $w' \leq 0$ on $(0, \phi_i]$, with $H_0 = \sqrt{\tilde{V}_i/3}/M$; the script checks it for $\lambda\sqrt{\phi}$, V_0 and $\lambda\phi^4$ (it holds in all five runs, e.g. $0.696 \geq 0.186$ for V_0 const, $\chi'_i = -6$). For the two finite-time cases the Jordan time converges ($t_J = 6.458$ at $\tilde{H} = -10^4$ and $t_s = 6.4594$ by extrapolation for $-\lambda\phi^2$), and the local exponent is that of Remark 4.8.

Table 2: Numerical illustrations [NUMERICAL]. (ϕ_i, χ'_i) are initial data in the Einstein frame. “Reflected” gives the minimum ϕ reached. Crunch times t_J are Jordan times at $\tilde{H} = -10^4$.

Potential (parameters)	(ϕ_i, χ'_i)	Outcome	Class
$-V_0 + \frac{1}{2}m^2\phi^2$ ($V_0=0.01$, $m^2=0.1$)	$(1, -1)$, $(1, -4)$ $(0.5, 0)$	reflected, $\phi_{\min} = 0.93, 0.75$ crunch, $\phi \rightarrow 0$ at $t_J \simeq 6.89$	outside inner basin Type I (num.)
$\lambda\sqrt{\phi}$ ($\lambda=0.05$)	$(1, -1)$, $(1, -6)$	reflected, $\phi_{\min} = 0.96, 0.78$	Regular, Thm (a)
$V_0 > 0$ const ($V_0=0.01$)	$(1, -1)$, $(1, -6)$	reflected, $\phi_{\min} = 0.90, 0.70$	Regular, Thm (a)
$\lambda\phi^4$ ($\lambda=0.01$)	$(1, -1)$	excursion bounded, $\phi_{\min} = 0.79$	Regular, Thm (a)
$\lambda\phi^4(1 - \phi^2/10)$ ($\lambda=0.01$)	$(1, -1)$, $(1, -6)$	reflected, $\phi_{\min} = 0.80, 0.54$	Regular, Thm (a) [†]
$V_0 e^{-1/\phi}$ ($V_0=0.1$)	$(1, -1)$, $(1, -8)$ $(0.1, -0.1)$	reflected, $\phi_{\min} = 0.95, 0.71$ monotone, $\phi = 0.0098$ at $\tau = 4000$	outside inner basin ($\phi > 1/4$) Type II, Thm (b)
$\lambda\phi^6$ ($\lambda=0.01$)	$(1, -1)$	monotone, $\phi = 4.2 \times 10^{-3}$ at $\tau = 4000$ ($t_J \simeq 1.2 \times 10^6$)	Type II, Thm (b)
$\lambda\phi^4(1 + \phi^2/2)$ ($\lambda=0.01$)	$(1, -1)$ $(1, -6)$	$\phi = 0.1307, 0.04161$ at $\tau = 10^3, 10^4$ $\phi = 0.1286, 0.04154$ at $\tau = 10^3, 10^4$	Type II, Thm (b) Type II, Thm (b)
$-\lambda\phi^2$ ($\lambda=0.01$)	$(1, -1.5)$ $(1, +1.5)$	crunch, $\phi \rightarrow 0$ at $t_J \simeq 6.459$ crunch at large ϕ ($\phi > 20$, growing; $t_J \simeq 5.80$)	Type I (num.) —
$-\lambda\phi^4$ ($\lambda=0.01$)	$(1, -1.5)$	crunch, $\phi \rightarrow 0$, $t_J \simeq 9.32$	Type I (num.)
$-\lambda\phi^n$, $n = 6, 8, 12, 13$	$(1, -2)$	crunch, $\phi \rightarrow 0$ ($t_J \simeq 18, 38, 1.9 \times 10^3, 3.7 \times 10^5$)	Type I (num.)
$-\lambda\phi^n$, $n = 14, 16$	$(1, -2)$	monotone, $\phi = 0.081, 0.076$ at $\tau = 3 \times 10^4$	asymptotic

[†] This test potential becomes negative for $\phi > \sqrt{10}$, outside the region explored by the trajectories above.

Remark 5.1 (Why Jordan-frame linear stability is not enough). A linearisation of (4) around $\phi = 0$ that keeps only $\ddot{\phi} + 3H\dot{\phi} + V''(0)\phi \simeq 0$ drops the curvature term $-\phi(2H^2 + \dot{H})$, which is not small near $\phi \rightarrow 0$ because H diverges there (Sec. 3). For $V = -\lambda\phi^2$ the naive linear problem suggests a repelling point, whereas in the Einstein frame $\tilde{V} = -36M^4\lambda/\phi^2 \rightarrow -\infty$ attracts the field: $\phi = 0$ is reached in finite time (Table 2). Finite-time Lyapunov exponents are likewise ill defined at a finite-time singularity where the Jacobian diverges; the criteria of this note are phrased through $\tilde{V}$ instead.

6 Implications for Quantum Distortion Gravity

6.1 In the Weyl-gauged theory the dilaton is pure gauge

In QDG the dilaton is coupled through $D_\mu\phi = (\partial_\mu - V_\mu)\phi$ and $\tilde{R} = \hat{R} - 6\hat{\nabla} \cdot V - 6V^2$ [1, 2] (Weyl weight -1 for ϕ ; the book’s convention is $D_\mu\Phi = (\partial_\mu + wV_\mu)\Phi$ with $w = -1$). Collecting the V_μ -dependent terms of $\frac{1}{12}\phi^2\tilde{R} - \frac{1}{2}(D\phi)^2$ gives $2\phi V \cdot \partial\phi - \phi^2 V^2$ (up to a total derivative), so with a vanishing field strength the V_μ equation of motion is

$$V_\mu = \partial_\mu \ln \phi, \quad (12)$$

which is pure gauge and gives $D_\mu\phi = 0$ and $\tilde{R} = (\phi/\phi_0)^2 R[\hat{g}]$ for $\hat{g} = (\phi/\phi_0)^2 g$. The action reduces to Einstein–Hilbert for $\hat{g}$, with $M_{\text{Pl}}^2 = \phi_0^2/6$, plus $-\lambda\phi_0^4\sqrt{-\hat{g}}$ for $V = \lambda\phi^4$: this is the unitary gauge of the monograph, and $\phi = 0$ is merely the locus where the Weyl transformation

to that gauge is singular. This agrees with the monograph’s statement that the dilaton is eaten by the Weyl gauge field and has a constant potential in the Einstein frame [2]. Two consequences follow. [DERIVED]

1. The equations of Sec. 2 describe the *ungauged* theory (1), not gauged QDG: a Jordan gauge with a dynamical $\phi(t)$ and $V_\mu = 0$ is inconsistent with the V_μ equation above, so the cosmological system of Sec. 2 must not be read as the cosmology of gauged QDG.
2. The only potential compatible with local Weyl invariance is $V = \lambda\phi^4$, since $\sqrt{-g}V(\phi)$ scales as Ω^{4-n} for $V \propto \phi^n$. For $n = 4$ one has $w \equiv \lambda$, hence $w' \equiv 0$ and $\tilde{V} = \text{const}$: by Theorem 4.3(a) this case is WISC-regular. Weyl invariance of the potential thus implies regularity; the point $\phi = 0$ becomes accessible only through explicit breaking, $w' \neq 0$. Any other $V(\phi)$ (e.g. Coleman–Weinberg corrections) breaks Weyl invariance explicitly, and the WISC analysis then applies to that breaking sector.

The Weyl-gauged sector beyond the dilaton (the axial vector, the non-minimal distortion couplings, the field strength of V_μ) is not included in this analysis, and its effect on the accessibility of $\phi = 0$ is open. [OPEN]

6.2 Relation to Weyl-lift singularity-resolution proposals

The Einstein-frame reading is relevant to the literature in which $\phi = 0$ is crossed into an “antigravity” region [4, 5]. In the reduction of Sec. 3, $\phi \rightarrow 0$ is at infinite field distance and the Einstein-frame evolution cannot continue through it; any continuation to $\phi < 0$ is therefore an additional prescription, not a property of the smooth Einstein-frame solution. We do not take a position on whether that prescription resolves the singularity; the cited literature itself discusses the behaviour of Weyl-invariant quantities at the transition. For $w \rightarrow \text{const}$ as $\phi \rightarrow 0$, $\tilde{V}$ tends to a plateau at $\chi \rightarrow -\infty$, which is structurally reminiscent of the plateau of α -attractor potentials [3]; we draw no further parallel.

7 Conclusions

The reduction to a canonical scalar in the Einstein frame turns the WISC question into a statement about $\tilde{V} = 36M^4V/\phi^4$ as $\chi \rightarrow -\infty$. The corrected WISC criterion, proved for $V > 0$ (Theorem 4.3), is

$$\text{regular} \iff w' \leq 0 \text{ near } 0, \quad \text{Type II} \iff w' > 0 \text{ near } 0, \quad w \equiv V/\phi^4,$$

that is, the conformal point is reached if and only if the explicit breaking of scale invariance pushes the field toward it. The naive rule (“singular if and only if $V \rightarrow 0$ ”) fails in both directions: $V \propto \phi^n$ with $0 < n \leq 4$ (including the mass term and the quartic) is regular although $V(0) = 0$. For $V < 0$ near the origin the numerical evidence is for finite-time crunches (Type I) with $\phi \propto (t_s - t)^{1/(3\sqrt{2}-1)}$ and $a_J \propto (t_s - t)^{(\sqrt{2}-1)/(3\sqrt{2}-1)}$, with the caveats of Sec. 4.3.

Open problems.

- [OPEN] A proof of the Type I statement for $V < 0$ (Theorem 4.3 covers $V > 0$ only), and the sharp recollapse threshold for $V = -\lambda\phi^n$.
- [OPEN] The full basin of initial data, in particular its dependence on the kinetic energy, when w' changes sign more than once, and for $V < 0$.
- [OPEN] Contracting initial branches, in which E is not monotone and a wall can in principle be overcome.

- [OPEN] The system with additional matter, for which the Einstein-frame map couples matter to ϕ .
- [OPEN] Non-analytic potentials, perturbations, and the Weyl-gauged sector of QDG (axial vector, field strength of V_μ , explicit Weyl-breaking terms).

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Author contributions

L. E. Pavesi: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft preparation, Writing – review and editing.

Additional information

During the preparation of this manuscript, large language models were used to assist with symbolic and numerical verification of the equations and with language editing. All scientific claims and conclusions are the author’s responsibility.

Conflicts of Interest

The author is the Editor-in-Chief of JAGP, the journal in which this article is published. This constitutes a structural conflict of interest. External anonymous peer review has been used to mitigate it. The author has recused himself from editorial handling.

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